Matrices

Introduction

Matrices - Introduction

Matrix algebra has at least two advantages:

•Reduces complicated systems of equations to simple expressions

•Adaptable to systematic method of mathematical treatment and well suited to computers

Definition:

A matrix is a set or group of numbers arranged in a square or rectangular array enclosed by two brackets

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 4 & 2 \\ -3 & 0 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Matrices - Introduction

Properties:

•A specified number of rows and a specified number of columns

•Two numbers (rows x columns) describe the dimensions or size of the matrix.

Examples:

3x3 matrix

2x4 matrix

1x2 matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & -1 & 5 \\ 3 & 3 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 3 & -3 \\ 0 & 0 & 3 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Matrices - Introduction

A matrix is denoted by a bold capital letter and the elements within the matrix are denoted by lower case letters

e.g. matrix [**A**] with elements a_{ij}

$$\mathbf{A}_{mxn} = \quad {}_m\mathbf{A}^n \quad \begin{bmatrix} a_{11} & a_{12} \ldots & a_{ij} & a_{in} \\ a_{21} & a_{22} \ldots & a_{ij} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{ij} & a_{mn} \end{bmatrix}$$

i goes from 1 to m

j goes from 1 to n

Matrices - Introduction

TYPES OF MATRICES

1. Column matrix or vector:

The number of rows may be any integer but the number of columns is always 1

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

Matrices - Introduction

TYPES OF MATRICES

2. Row matrix or vector

Any number of columns but only one row

$$\begin{bmatrix} 1 & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \end{bmatrix}$$

Matrices - Introduction

TYPES OF MATRICES

3. Rectangular matrix

Contains more than one element and number of rows is not equal to the number of columns

$$\begin{bmatrix} 1 & 1 \\ 3 & 7 \\ 7 & -7 \\ 7 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 3 & 0 \end{bmatrix}$$

$$m \neq n$$

Matrices - Introduction

TYPES OF MATRICES

4. Square matrix

The number of rows is equal to the number of columns

(a square matrix $\mathbf{A}_{m \times m}$ has an order of m)

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 9 & 9 & 0 \\ 6 & 6 & 1 \end{bmatrix}$$

The principal or main diagonal of a square matrix is composed of all elements a_{ij} for which $i=j$

Matrices - Introduction

TYPES OF MATRICES

5. Diagonal matrix

A square matrix where all the elements are zero except those on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$

$a_{ij} \neq 0$ for some or all $i = j$

Matrices - Introduction

TYPES OF MATRICES

6. Unit or Identity matrix - I

A diagonal matrix with ones on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$

$a_{ij} = 1$ for some or all $i = j$

Matrices - Introduction

TYPES OF MATRICES

7. Null (zero) matrix - 0

All elements in the matrix are zero

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$a_{ij} = 0$$ For all i,j

Matrices - Introduction

TYPES OF MATRICES

8. Triangular matrix

A square matrix whose elements above or below the main diagonal are all zero

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

Matrices - Introduction

TYPES OF MATRICES

8a. Upper triangular matrix

A square matrix whose elements below the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & a_{ij} & a_{ij} \\ 0 & a_{ij} & a_{ij} \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 7 & 4 & 4 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i > j$

Matrices - Introduction

TYPES OF MATRICES

8b. Lower triangular matrix

A square matrix whose elements above the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ a_{ij} & a_{ij} & 0 \\ a_{ij} & a_{ij} & a_{ij} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i < j$

Matrices – Introduction

TYPES OF MATRICES

9. Scalar matrix

A diagonal matrix whose main diagonal elements are equal to the same scalar

A scalar is defined as a single number or constant

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ 0 & a_{ij} & 0 \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$
$a_{ij} = a$ for all $i = j$

Matrices

Matrix Operations

Matrices - Operations

EQUALITY OF MATRICES

Two matrices are said to be equal only when all corresponding elements are equal

Therefore their size or dimensions are equal as well

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \mathbf{A} = \mathbf{B}$$

Matrices - Operations

Some properties of equality:

•IIf **A** = **B**, then **B** = **A** for all **A** and **B**

•IIf **A** = **B**, and **B** = **C**, then **A** = **C** for all **A**, **B** and **C**

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

If **A** = **B** then $a_{ij} = b_{ij}$

Matrices - Operations

ADDITION AND SUBTRACTION OF MATRICES

The sum or difference of two matrices, **A** and **B** of the same size yields a matrix **C** of the same size

$$c_{ij} = a_{ij} + b_{ij}$$

Matrices of different sizes cannot be added or subtracted

Matrices - Operations

Commutative Law:

A + **B** = **B** + **A**

Associative Law:

A + (**B** + **C**) = (**A** + **B**) + **C** = **A** + **B** + **C**

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix}$$

A	**B**	**C**
2x3	2x3	2x3

Matrices - Operations

$\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$

$\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$ (where $-\mathbf{A}$ is the matrix composed of $-a_{ij}$ as elements)

$$\begin{bmatrix} 6 & 4 & 2 \\ 3 & 2 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

Matrices - Operations

SCALAR MULTIPLICATION OF MATRICES

Matrices can be multiplied by a scalar (constant or single element)

Let k be a scalar quantity; then

kA = Ak

Ex. If k=4 and

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix}$$

Matrices - Operations

$$4 \times \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix} \times 4 = \begin{bmatrix} 12 & -4 \\ 8 & 4 \\ 8 & -12 \\ 16 & 4 \end{bmatrix}$$

Properties:

- k (**A** + **B**) = k**A** + k**B**
- (k + g)**A** = k**A** + g**A**
- k(**AB**) = (k**A**)**B** = **A**(k)**B**
- k(g**A**) = (kg)**A**

Matrices - Operations

MULTIPLICATION OF MATRICES

The product of two matrices is another matrix

Two matrices **A** and **B** must be **conformable** for multiplication to be possible

i.e. the number of columns of **A** must equal the number of rows of **B**

Example.

$$\mathbf{A} \quad x \quad \mathbf{B} \quad = \quad \mathbf{C}$$

$$(1x3) \quad (3x1) \quad (1x1)$$

Matrices - Operations

B x **A** = Not possible!

(2x1) (4x2)

A x **B** = Not possible!

(6x2) (6x3)

Example

A x **B** = **C**

(2x3) (3x2) (2x2)

Matrices - Operations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$(a_{11} \times b_{11}) + (a_{12} \times b_{21}) + (a_{13} \times b_{31}) = c_{11}$$
$$(a_{11} \times b_{12}) + (a_{12} \times b_{22}) + (a_{13} \times b_{32}) = c_{12}$$
$$(a_{21} \times b_{11}) + (a_{22} \times b_{21}) + (a_{23} \times b_{31}) = c_{21}$$
$$(a_{21} \times b_{12}) + (a_{22} \times b_{22}) + (a_{23} \times b_{32}) = c_{22}$$

Successive multiplication of row *i* of **A** with column *j* of **B** – row by column multiplication

Matrices - Operations

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 7 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} (1\times 4)+(2\times 6)+(3\times 5) & (1\times 8)+(2\times 2)+(3\times 3) \\ (4\times 4)+(2\times 6)+(7\times 5) & (4\times 8)+(2\times 2)+(7\times 3) \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Remember also:

IA = **A**

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix} = \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Matrices - Operations

Assuming that matrices **A**, **B** and **C** are conformable for the operations indicated, the following are true:

1. **AI** = **IA** = **A**
2. **A**(**BC**) = (**AB**)**C** = **ABC** - (associative law)
3. **A**(**B**+**C**) = **AB** + **AC** - (first distributive law)
4. (**A**+**B**)**C** = **AC** + **BC** - (second distributive law)

Caution!

1. **AB** not generally equal to **BA**, **BA** may not be conformable
2. If **AB** = **0**, neither **A** nor **B** necessarily = **0**
3. If **AB** = **AC**, **B** not necessarily = **C**

Matrices - Operations

AB not generally equal to **BA**, **BA** may not be conformable

$$T = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$$

$$TS = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 15 & 20 \end{bmatrix}$$

$$ST = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 23 & 6 \\ 10 & 0 \end{bmatrix}$$

Matrices - Operations

If **AB** = **0**, neither **A** nor **B** necessarily = **0**

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Matrices - Operations

TRANSPOSE OF A MATRIX

If :

$$\underset{2\text{x}3}{A} = {}_2A^3 = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

Then transpose of A, denoted A^T is:

$$A^T = {}_2A^{3^T} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$

$$a_{ij} = a_{ji}^T$$ For all *i* and *j*

Matrices - Operations

To transpose:

Interchange rows and columns

The dimensions of $\mathbf{A}^T$ are the reverse of the dimensions of $\mathbf{A}$

$$A = {}_2A^3 = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

2 x 3

$$A^T = {}_3A^{T\,2} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$

3 x 2

Matrices - Operations

Properties of transposed matrices:

1. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
2. $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$
3. $(k\mathbf{A})^T = k\mathbf{A}^T$
4. $(\mathbf{A}^T)^T = \mathbf{A}$

Matrices - Operations

1. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 2 \\ 3 & -5 \\ -1 & 6 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 5 & -2 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

Matrices - Operations

$(\mathbf{AB})^T = \mathbf{B}^T\ \mathbf{A}^T$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 8 \end{bmatrix}$$

Matrices - Operations

SYMMETRIC MATRICES

A Square matrix is symmetric if it is equal to its transpose:

$$\mathbf{A} = \mathbf{A}^{\mathrm{T}}$$

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

$$A^T = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

Matrices - Operations

When the original matrix is square, transposition does not affect the elements of the main diagonal

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

The identity matrix, **I**, a diagonal matrix **D**, and a scalar matrix, **K**, are equal to their transpose since the diagonal is unaffected.

Matrices - Operations

INVERSE OF A MATRIX

Consider a scalar k. The inverse is the reciprocal or division of 1 by the scalar.

Example:

k=7   the inverse of k or $k^{-1} = 1/k = 1/7$

Division of matrices is not defined since there may be **AB** = **AC** while $\mathbf{B} \neq \mathbf{C}$

Instead matrix inversion is used.

The inverse of a square matrix, **A**, if it exists, is the unique matrix $\mathbf{A}^{-1}$ where:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Matrices - Operations

Example:

$$A =_{2} A^{2} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Because:

$$\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrices - Operations

Properties of the inverse:

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(A^{-1})^{-1} = A$$

$$(A^{T})^{-1} = (A^{-1})^{T}$$

$$(kA)^{-1} = \frac{1}{k}A^{-1}$$

A square matrix that has an inverse is called a nonsingular matrix

A matrix that does not have an inverse is called a singular matrix

Square matrices have inverses except when the determinant is zero

When the determinant of a matrix is zero the matrix is singular

Matrices - Operations

DETERMINANT OF A MATRIX

To compute the inverse of a matrix, the determinant is required

Each square matrix **A** has a unit scalar value called the determinant of **A**, denoted by det **A** or |**A**|

If $$A = \begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix}$$

then $$|A| = \begin{vmatrix} 1 & 2 \\ 6 & 5 \end{vmatrix}$$

Matrices - Operations

If **A** = [**A**] is a single element (1x1), then the determinant is defined as the value of the element

Then |**A**| =det **A** = a_{11}

If **A** is (n x n), its determinant may be defined in terms of order (n-1) or less.

Matrices - Operations

MINORS

If **A** is an n x n matrix and one row and one column are deleted, the resulting matrix is an (n-1) x (n-1) submatrix of **A**.

The determinant of such a submatrix is called a minor of **A** and is designated by m_{ij} , where *i* and *j* correspond to the deleted

row and column, respectively.

m_{ij} is the minor of the element a_{ij} in **A**.

Matrices - Operations

eg.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Each element in **A** has a minor

Delete first row and column from **A** .

The determinant of the remaining 2 x 2 submatrix is the minor of a_{11}

$$m_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Matrices - Operations

Therefore the minor of a_{12} is:

$$m_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

And the minor for a_{13} is:

$$m_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Matrices - Operations

COFACTORS

The cofactor C_{ij} of an element a_{ij} is defined as:

$$C_{ij} = (-1)^{i+j} m_{ij}$$

When the sum of a row number *i* and column *j* is even, $c_{ij} = m_{ij}$ and when *i*+*j* is odd, $c_{ij} = -m_{ij}$

$$c_{11}(i=1, j=1) = (-1)^{1+1} m_{11} = +m_{11}$$

$$c_{12}(i=1, j=2) = (-1)^{1+2} m_{12} = -m_{12}$$

$$c_{13}(i=1, j=3) = (-1)^{1+3} m_{13} = +m_{13}$$

Matrices - Operations

DETERMINANTS CONTINUED

The determinant of an n x n matrix **A** can now be defined as

$$|A| = \det A = a_{11}c_{11} + a_{12}c_{12} + \ldots + a_{1n}c_{1n}$$

The determinant of **A** is therefore the sum of the products of the elements of the first row of **A** and their corresponding cofactors.

(It is possible to define |**A**| in terms of any other row or column but for simplicity, the first row only is used)

Matrices - Operations

Therefore the 2 x 2 matrix :

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Has cofactors :

$$c_{11} = m_{11} = |a_{22}| = a_{22}$$

And:

$$c_{12} = -m_{12} = -|a_{21}| = -a_{21}$$

And the determinant of **A** is:

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Matrices - Operations

Example 1:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$|A| = (3)(2) - (1)(1) = 5$$

Matrices - Operations

For a 3 x 3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The cofactors of the first row are:

$$c_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

$$c_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{23}a_{31})$$

$$c_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$$

Matrices - Operations

The determinant of a matrix A is:

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Which by substituting for the cofactors in this case is:

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Matrices - Operations

Example 2:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$|A| = (1)(2-0) - (0)(0+3) + (1)(0+2) = 4$$

Matrices - Operations

ADJOINT MATRICES

A cofactor matrix **C** of a matrix **A** is the square matrix of the same order as **A** in which each element a_{ij} is replaced by its cofactor c_{ij} .

Example:

If $$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

The cofactor C of A is $$C = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$$

Matrices - Operations

The adjoint matrix of **A**, denoted by adj **A**, is the transpose of its cofactor matrix

$$adjA = C^T$$

It can be shown that:

A(adj **A**) = (adj**A**) **A** = |**A**| **I**

Example:

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$|A| = (1)(4) - (2)(-3) = 10$$

$$adjA = C^T = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$$

Matrices - Operations

$$A(adjA) = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10I$$

$$(adjA)A = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10I$$

Matrices - Operations

USING THE ADJOINT MATRIX IN MATRIX INVERSION

Since

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

and

$$\mathbf{A}(\text{adj }\mathbf{A}) = (\text{adj}\mathbf{A})\,\mathbf{A} = |\mathbf{A}|\,\mathbf{I}$$

then

$$A^{-1} = \frac{adjA}{|A|}$$

Matrices - Operations

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}$$

To check

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^{-1}A = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Matrices - Operations

Example 2

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

The determinant of **A** is

|**A**| = (3)(-1-0)-(-1)(-2-0)+(1)(4-1) = -2

The elements of the cofactor matrix are

$$c_{11} = +(-1), \qquad c_{12} = -(-2), \qquad c_{13} = +(3),$$

$$c_{21} = -(-1), \qquad c_{22} = +(-4), \qquad c_{23} = -(7),$$

$$c_{31} = +(-1), \qquad c_{32} = -(-2), \qquad c_{33} = +(5),$$

Matrices - Operations

The cofactor matrix is therefore

$$C = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -4 & -7 \\ -1 & 2 & 5 \end{bmatrix}$$

so

$$adjA = C^T = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix}$$

and

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix}$$

Matrices - Operations

The result can be checked using

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

The determinant of a matrix must not be zero for the inverse to exist as there will not be a solution

Nonsingular matrices have non-zero determinants

Singular matrices have zero determinants

Matrix Inversion

Simple 2 x 2 case

Simple 2 x 2 case

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and

$$A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

Since it is known that

$$\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$$

then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Simple 2 x 2 case

Multiplying gives

$$aw + by = 1$$

$$ax + bz = 0$$

$$cw + dy = 0$$

$$cx + dz = 1$$

It can simply be shown that

$$|A| = ad - bc$$

Simple 2 x 2 case

thus

$$y = \frac{1 - aw}{b}$$

$$y = \frac{-cw}{d}$$

$$\frac{1 - aw}{b} = \frac{-cw}{d}$$

$$w = \frac{d}{da - bc} = \frac{d}{|A|}$$

Simple 2 x 2 case

$$z = \frac{-ax}{b}$$

$$z = \frac{1-cx}{d}$$

$$\frac{-ax}{b} = \frac{1-cx}{d}$$

$$x = \frac{b}{-da+bc} = -\frac{b}{|A|}$$

Simple 2 x 2 case

$$w = \frac{1 - by}{a}$$

$$w = \frac{-dy}{c}$$

$$\frac{1 - by}{a} = \frac{-dy}{c}$$

$$y = \frac{c}{-ad + cb} = -\frac{c}{|A|}$$

Simple 2 x 2 case

$$x = \frac{-bz}{a}$$

$$x = \frac{1-dz}{c}$$

$$\frac{-bz}{a} = \frac{1-dz}{c}$$

$$z = \frac{a}{ad-bc} = \frac{a}{|A|}$$

Simple 2 x 2 case

So that for a 2 x 2 matrix the inverse can be constructed in a simple fashion as

$$A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} \dfrac{d}{|A|} & \dfrac{b}{|A|} \\ \dfrac{-c}{|A|} & \dfrac{a}{|A|} \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Exchange elements of main diagonal
- Change sign in elements off main diagonal
- Divide resulting matrix by the determinant

Simple 2 x 2 case

Example

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

$$A^{-1} = -\frac{1}{10}\begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -0.1 & 0.3 \\ 0.4 & -0.2 \end{bmatrix}$$

Check inverse

$\mathbf{A}^{-1}\,\mathbf{A}=\mathbf{I}$

$$-\frac{1}{10}\begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Matrices and Linear Equations

Linear Equations

Linear Equations

Linear equations are common and important for survey problems

Matrices can be used to express these linear equations and aid in the computation of unknown values

Example

n equations in n unknowns, the a_{ij} are numerical coefficients, the b_i are constants and the x_j are unknowns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

Linear Equations

The equations may be expressed in the form

AX = **B**

where

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n1} \cdots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

n x n   n x 1   n x 1

Number of unknowns = number of equations = n

Linear Equations

If the determinant is nonzero, the equation can be solved to produce n numerical values for x that satisfy all the simultaneous equations

To solve, premultiply both sides of the equation by $\mathbf{A}^{-1}$ which exists because $|\mathbf{A}| \neq \mathbf{0}$

$$\mathbf{A}^{-1}\,\mathbf{A}\mathbf{X} = \mathbf{A}^{-1}\,\mathbf{B}$$

Now since

$$\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

We get

$$\mathbf{X} = \mathbf{A}^{-1}\,\mathbf{B}$$

So if the inverse of the coefficient matrix is found, the unknowns, **X** would be determined

Linear Equations

Example

$$3x_1 - x_2 + x_3 = 2$$

$$2x_1 + x_2 = 1$$

$$x_1 + 2x_2 - x_3 = 3$$

The equations can be expressed as

$$\begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Linear Equations

When $\mathbf{A}^{-1}$ is computed the equation becomes

$$X = A^{-1}B = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}$$

Therefore

$$x_1 = 2,$$
$$x_2 = -3,$$
$$x_3 = -7$$

Linear Equations

The values for the unknowns should be checked by substitution back into the initial equations

$$x_1 = 2,$$
$$x_2 = -3,$$
$$x_3 = -7$$

$$3x_1 - x_2 + x_3 = 2$$
$$2x_1 + x_2 = 1$$
$$x_1 + 2x_2 - x_3 = 3$$

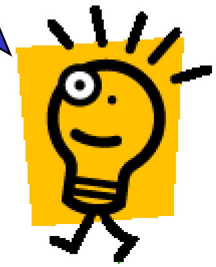
$$3 \times (2) - (-3) + (-7) = 2$$
$$2 \times (2) + (-3) = 1$$
$$(2) + 2 \times (-3) - (-7) = 3$$

Complex Numbers

Lesson 5.1

The Imaginary Number *i*

It's any number you can imagine



- By definition

$$\sqrt{-1} = i \quad \Leftrightarrow \quad i^2 = -1$$

- Consider powers if *i*

$$i^2 = -1$$

$$i^3 = i^2 \cdot i = -i$$

$$i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$$

$$i^5 = i^4 \cdot i = 1 \cdot i = i$$

$$\ldots$$

Using ***i***

- Now we can handle quantities that occasionally show up in mathematical solutions

$$\sqrt{-a} = \sqrt{-1} \cdot \sqrt{a} = i\sqrt{a}$$

- What about $\sqrt{-49}$ $\sqrt{-18}$

Complex Numbers

- Combine real numbers with imaginary numbers
  - a + b***i***

Real part

Imaginary part

- Examples

$$3 + 4i$$

$$-6 + \frac{3}{2}i$$

$$4.5 + i \cdot 2\sqrt{6}$$

Try It Out

- Write these complex numbers in standard form   a + b***i***

$$9-\sqrt{-75}$$

$$\sqrt{-16}+7$$

$$5-\sqrt{-144}$$

$$-\sqrt{-100}$$

Operations on Complex Numbers

- Complex numbers can be combined with
  - addition
  - subtraction
  - multiplication
  - division
- Consider

$$(-3+i)-(-8+2i)$$

$$(9-12i)\cdot(7+15i)$$

$$(2-4i)+(4+3i)$$

Operations on Complex Numbers

- Division technique
  - Multiply numerator and denominator by the conjugate of the denominator

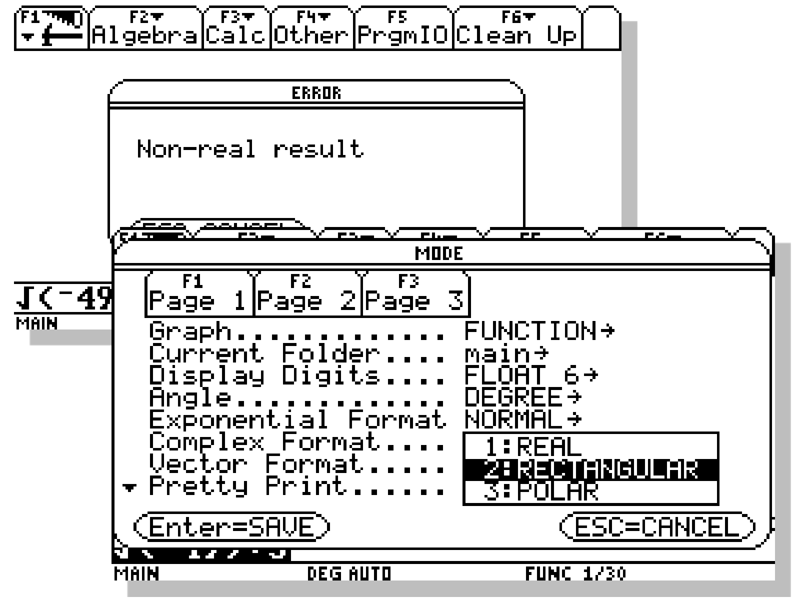
$$\frac{3i}{5-2i} = \frac{3i}{5-2i}\cdot\frac{5+2i}{5+2i}$$

$$= \frac{15i+6i^2}{25-4i^2}$$

$$= \frac{-6+15i}{29} = -\frac{6}{29}+\frac{15}{29}i$$

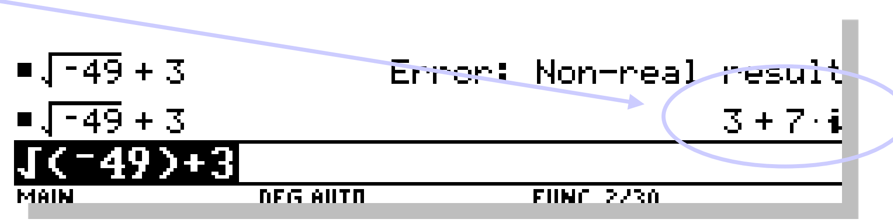
Complex Numbers on the Calculator

- Possible result



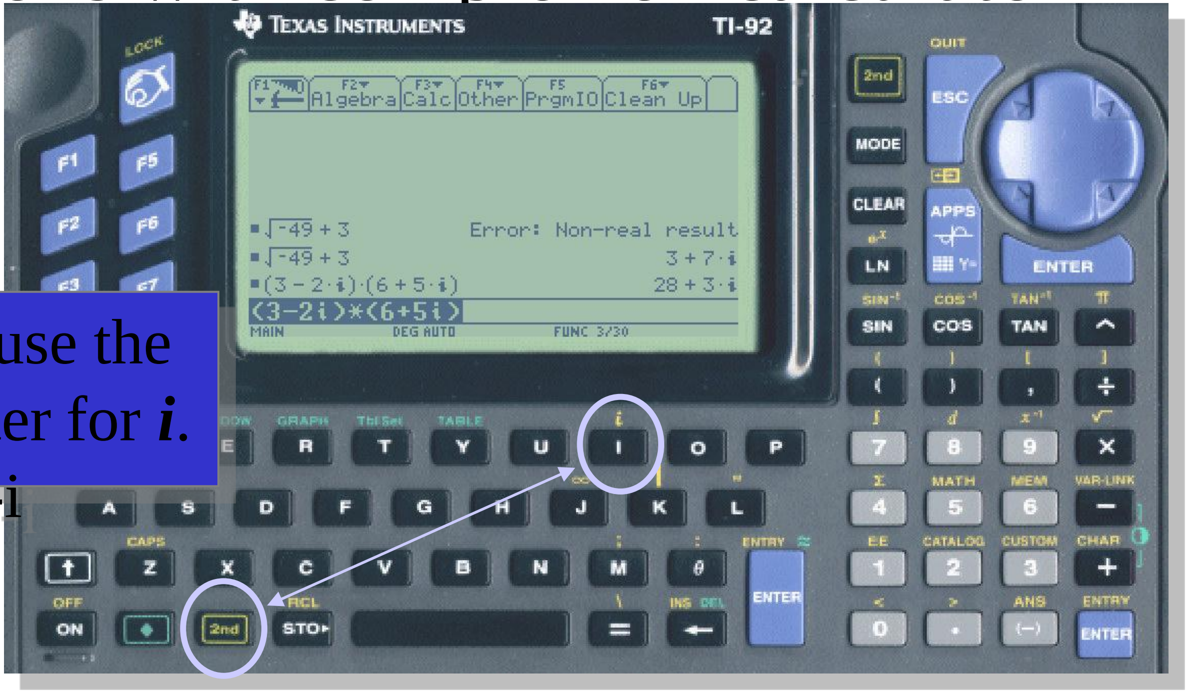
- Reset mode Complex format to <u>Rectangular</u>

- Now calculator does desired result



Complex Numbers on the Calculator

- Operations with complex on calculator



Make sure to use the correct character for ***i***.

Use 2^{nd}-i

Warning

- Consider $\sqrt{-16}\cdot\sqrt{-49}$

- It is tempting to combine them

$$\sqrt{-16\cdot -49} = \sqrt{+16\cdot 49} = 4\cdot 7 = 28$$

  - The multiplicative property of radicals only works for positive values under the radical sign
  - Instead use imaginary numbers

$$\sqrt{-16\cdot -49} = 4i\cdot 7i = 4\cdot 7\cdot i^2 = -28$$

Try It Out

- Use the correct principles to simplify the following:

$$\sqrt{-3}\cdot\sqrt{-121}$$

$$\left(4+\sqrt{-81}\right)\cdot\left(4-\sqrt{-81}\right)$$

$$\left(3-\sqrt{-144}\right)^2$$

Assignment

- Lesson 5.1
- Page 340
- Exercises 1 – 69 EOO

STATISTICS

Definations: Statistics ; Measure of central tendency

- <u>Defination of statistics</u>: statistics may be defined as the science of collection, organization, analysis and interpretation of numerical data.
- <u>Measures of Central Tendency</u>: An average is called a measure of central tendency, because it tends to lie centrally with the values of the variable arranged according to magnitude.

Arithmetic Mean(A.M.):

The arithmetic mean of an individual series is defined as the quotient of the sum of all the values of the variable by the total number of items.

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Example 1

Example : The blood pressure of seven middle aged men were as follows:

151, 124, 132, 170, 146, 124 and 113.

$$\bar{x} = \frac{(151+124+132+170+146+124+113)}{7}$$

The mean is $= 137.14$

In case of discrete frequency distribution A.M. is calculated as:

$$\text{A.M. } (\bar{X}) = \sum(f_i\ x_i) / \sum f_i$$

$$\text{OR } \sum(f\,x) / \sum f$$

Where f_i is the frequency of x_i $(1 \leq I \leq n)$

Example2

Calculate the A.M. for the following data:

Income (in rs.): 500 520 550 600 800 1000

No. of emp: 4 10 6 5 3 2

Sol.

S.no.	Income in rs. (x_i)	No. of employees (f_i)	(f_i x_i)
1.	500	4	2000
2.	520	10	5200
3.	550	6	3300
4.	600	5	3000
5.	800	3	2400
6.	1000	2	2000
		$\sum f = 30$	$\sum(f_i\ x_i) =$ 17900

Mean $= \sum(f_i\ x_i) / \sum f_i = 17900/30 = 596.67$

For Grouped or continuous frequency distribution, Arithmatic mean is calculated as:

$$\text{Arithmatic Mean} = A + \frac{\sum f_i u_i}{\sum f_i} * h$$

This method is called ***STEP DEVIATION METHOD***

EXAMPLE 3

Calculate the Arithmetic mean of the marks scored by the students of a class in a class test from the following data :

Marks	*Number of students*	*Mid Point (x_i)*	*ui= (xi – A)/h*	*fiui*
0-10	12	5	-2	-24
10-20	18	15	-1	-18
20-30	27	A = 25	0	0
30-40	20	35	1	20
40-50	17	45	2	34
50-60	6	55	3	18
Total	100			30

By step Deviation method, Arithmatic mean $= A+(\sum fi\ ui / \sum fi) * h$

$= 25 + (30/100)*10 = 28$

Median :

The median of a statistical series is defined as the size of the middle most item (or the A.M. of two middle most items), provided the items are in the order of magnitude.

<u>For an individual series, to find median we proceed as follow:</u>

(a) Arrange the observations in ascending or descending order of magnitude.

(b) <u>If n is odd</u>; then median = $(n+1/2)^{th}$ observation.

<u>If n is even</u>; them median = A.M. of $(n/2)^{th}$ & $(n/2) + 1^{th}$ observation.

Example 1:

Find the median of the values:

31 38 27 28 36 25 35 40

Sol: We arrange the values in ascending order

25 27 28 31 35 36 38 40

n = 8 (even);

Median = A.M. of $(8/2)^{th}$ & $(8/2) + 1^{th}$ observations

= (31+35)/2 = 33

In case of discrete frequency distribution Median is calculated as:

- Step I: Find the cumulative frequency (C.F.).
- Step 2: Find N/2 where N= $\sum f_i$
- Step 3: See the C.F. just greater than N/2.
- Step 4: The value of x corresponding to C.F. just greater than N/2 is median.

Example 2:

Calculate the Median of the following frequency distribution :

x	f	c.f.
1	8	8
2	10	18
3	11	29
4	16	45
5	20	65
6	25	90
7	15	105
8	9	114
9	6	120
	N = 120	

Here N is 120; N/2 is 60; C.F. just greater than N/2 is 65; So corresponding value of x '5' is median.

For Grouped or continuous frequency distribution, Median is calculated as:

- Step I: Find the cumulative frequency (C.F.).
- Step 2: Find N/2 where $N = \sum f_i$
- Step 3: The class corresponding to C.F. just greater than N/2 is median class and the value of the median is calculated by formula:

<u>Median = l + ((N/2-F)/f) *h</u>

Where l= lower limit of median class

F = C.F. of class preceding the median class

f is the frequency of median class

h is the width of median class.

Example 3

Calculate the median from the following distribution:

Class	Freuency	Cumative frequency
5-10	5	5
10-15	6	11 (F)
15-20 (Median class)	15 (f)	26 (C.F. just greater than N/2)
20-25	10	36
25-30	5	41
30-35	4	45
35-40	2	47
40-45	2	49
	N= 49	
Here N = 49	N/2 = 24.5	C.F. just greater than N/2 is 26
Corresponding class 15-20 is median class		Median = l +((N/2-F)/f) *h
Median = 15 + (24.5-11)/15 * 5 = 19.5		

MODE

- The mode of a distribution of the variable is that value of the variable for which the frequency is maximum.
- <u>In case of an individual series</u>, mode is calculated as by counting the number of times the various values repeat themselves and the value which occurs maximum no. of times is the modal value.

Example 1

Find the Mode of the following data:

110 120 130 120 110 140 130 120 140

120

Sol: Since the value 120 occurs the maximum no. of times. Hence the modal value is 120.

In case of discrete frequency distribution mode is calculated as:

For discrete frequency distribution, generally mode is calculated by finding the value for which frequency is maximum.

Example 2

Find the mode of the following distribution:

Size in inches	No of shirts sold
30	8
32	17
34	30
36	35
38	18
40	7
42	3

35 shirts of size 36 have the maximum sale. So mode of distribution is “36”.

For grouped or continuous frequency distribution mode is calculates as:

To find the mode of continuous frequency distribution, we follow the following steps:

STEP 1: Determine the class of maximum frequency, this class is modal class.

STEP 2: Determine the value of mode by applying the formula:

Mode $= l + (f - f_1 / f - f_1 - f_2) * h$

Where l is the lower limit of modal class
f is the frequency of modal class
h is the width of modal class
f_1 is the frequency of class preceding the modal class
f_2 is the frequency of class following the modal class

Example 3

Calculate the mode from the following data:

Rent (in Rs.)	No. of houses
20-40	6
40-60	9
60-80	11
80-100	14 (f_1)
100-120 (Modal class)	20 (f)
120-140	15 (f_2)
140-160	10
Highest frequency is 20	Hence Modal class is 100-120
Mode = $l + (f-f_1 / f-f_1-f_2) * h$	
Here l= 100; h= 20; f= 20; f_1 = 14; f_2 = 15	
Mode = 100 + (20-14)/(2*20-14-15) *20 = 110.91	

Measures of dispersion

- Mean Deviation: It is defined as the A.M. of the absolute deviations of all the values taken from any central value.
- Standard Deviation: The standard deviation of a statistical data is defined as the positive square root of the A.M. of the squared deviations of items from the A.M. of the series under consideration.

For individual series Mean deviation is calculated as:

$$\text{Mean Deviation } (\bar{x}) = \frac{\sum |x_i - \bar{x}|}{n}$$

$$\text{Mean Deviation (Median)} = \frac{\sum |x_i - \text{Median}|}{n}$$

$$\text{Cofficient of Mean Deviation } (\bar{x}) = \frac{\text{M.D.}(\bar{X})}{\bar{x}}$$

Cofficient of Mean Deviation (Median) = M.D.(Medain)/Median

Example 1

Calculate the mean deviation about mean and its coefficient for the following data:
21 23 25 28 30 32 46 38 48 46

S.No.	Xi	$xi - \bar{x}$	$\lvert x_i - \bar{x} \rvert$
1	21	-12.7	12.7
2	23	-10.7	10.7
3	25	-8.7	8.7
4	28	-5.7	5.7
5	30	-3.7	3.7
6	32	-1.7	1.7
7	46	12.3	12.3
8	38	4.3	4.3
9	48	14.3	14.3
10	46	12.3	12.3
Mean = 337/10 =33.7	M.D. = 86.4/10 = 8.64	Coeff. = 8.64/33.7 = 26	$\sum \lvert x_i - \bar{x} \rvert = 86.4$

For frequency distribution & Grouped Data M.D.is calculated as:

$$\text{Mean Deviation } (\bar{x}) = \frac{\sum f_i \left| x_i - \bar{x} \right|}{n}$$

$$\text{Mean Deviation (Median)} = \frac{\sum f_i \left| x_i - \text{Median} \right|}{N}$$

Example 2

Calculate the mean deviation about mean and its coefficient for the following frequency distribution:

X	f	fx	$X-\bar{x}$	$\lvert x - \bar{x} \rvert$	$f\lvert x - \bar{x} \rvert$
5	8	40	-4	4	32
7	6	42	-2	2	12
9	2	18	0	0	0
10	2	20	1	1	2
12	2	24	3	3	6
15	6	90	6	6	36
	N = 26	$\sum f x = 234$			$\sum f\lvert x - \bar{x} \rvert = 88$
Mean = 234/26=9		M.D. = 88/26= 3.38		Coefficient of M.D. = 3.38/9= 0.38	

Note:

- ***Same method will be used for finding the mean deviation about median.***

- ***Instead of mean, we are to find median first then find mean deviation by following the same procedure and same formulas.***

For Individual series & frequency distribution Standard Deviation is calculated as:

S.D.(Individual series) = $\sqrt{\sum_{i=1}^{i=n}(x\,i - \bar{x})^2 / n}$

Where x_i are the values of variable under consideration.

S.D. (For Discrete Frequency distribution):

$$\sqrt{\sum_{i=1}^{i=n} f i \,(x\,i - \bar{x})^2 / N}$$

Where N is the sum of all the frequencies.

Example 1

Find the S.D. and C.V. for the following data:
4,6,10,12,18

S.No.	x	$x - \bar{x}$	$(x - \bar{x})2$
1	4	-6	36
2	6	-4	16
3	10	0	0
4	12	2	4
5	18	8	x64
Mean = ∑x / n		Mean = 50/5 = 10	
S.D. = $\sqrt{\sum_{i=1}^{i=n}(x\,i - mean)^2 /n}$		S.D. = √120/5 = 4.899	
C.V. = (4.899/10) * 100 = 48.99 %			

Example 2

Calculate the S.D. and C.V. for the following

X	f	fx	$X - \bar{x}$	$(x - \bar{x})^2$	$f\,(x - \bar{x})^2$
5	12	60	-23	529	6348
15	18	270	-13	169	3042
25	27	675	-3	9	243
35	20	700	7	49	980
45	17	765	17	289	4913
55	6	330	27	729	4374
	$\sum f = 100$	$\sum fx = 2800$			$\sum f\,(x - \bar{x})^2 = 19900$

Mean = 2800/100 = 28

S.D. = $\sqrt{\sum_{i=1}^{i=n} f i\,(x i - \bar{x})^2}$ **/N** **S.D. = √ 19900/100 = 14.11**

C.V. = S.D. / Mean * 100 = 50.39%

For Grouped data S.D. is calculated as:

S.D. $=\sqrt{\sum f_i (u_i)^2 /N - (\sum f_i u_i /N)^2} /N * h$

Where $u_i = \dfrac{x_i - A}{h}$ and N is the sum of frequency.

Example 3

Find the S.D. & C.V. for the following data:

class	f	x	U = x-A/h	fu	u^2	fu^2
0-5	20	2.5	-4	-80	16	320
5-10	24	7.5	-3	-72	9	216
10-15	32	12.5	-2	-64	4	128
15-20	28	17.5	-1	-28	1	28
20-25	20	22.5	0	0	0	0
25-30	16	27.5	1	16	1	16
30-35	34	32.5	2	68	4	136
35-40	10	37.5	3	30	9	90
40-45	16	42.5	4	64	16	256
	$\sum f=200$			$\sum fu = -66$		$\sum fu^2 = 1190$

Mean = A + $\sum fu/\sum f * h$ = 20.85 S.D. = $\sqrt{\sum f_i (u_i)^2 /N - (\sum f_i u_i /N)^2} /N * h$ = 12.1

C.V. = C.V. = S.D. / Mean * 100 = 58.03%

OTHER FORMULAS (S.D.)

- Cofficient of S.D. = S.D./Mean

- Cofficient of variation = (S.D./Mean) *100

- Variance = Square of S.D.

Rank Correlation Coefficient:

Rank Coorelation Coefficient is given by the formula:

$$r = 1 - \frac{6 \sum d^2}{n\,(n^2 - 1)}$$

Where n is no. of items

d is the difference of ranks.

Example 1

Find the coefficient of rank correlation for the following data:

x	y	R1	R2	d=R1-R2	d^2
56	15	3	2	1	1
58	12	4	1	3	9
62	16	5	3	2	4
72	17	7	4	3	9
54	19	2	5	-3	9
52	20	1	6	-5	25
71	21	6	7	-1	1
90	22	9	8	1	1
81	23	8	9	-1	1
92	24	10	10	0	0
					$\sum d^2 = 60$

r = 1 – (6* 60)/10(100-1) = 0.64

Example 2

Seven Competitors in a music competition are ranked by the judges x & y in the following order. What is the degree of agreement between the judges. Also find the coefficient of correlation.

S.No.	Competitors	R1	R2	d = R1-R2	d^2
1	A	2	1	1	1
2	B	1	3	-2	4
3	C	4	2	2	4
4	D	3	4	-1	1
5	E	5	5	0	0
6	F	7	6	1	1
7	G	6	7	-1	1
					$\sum d^2 = 12$

r = 1-(72/7*48) = .7857 (Agreement between the judges is high)

Binomial Theorem

Session Objective

1. **Binomial theorem for positive integral index**
2. **Binomial coefficients — Pascal's triangle**
3. **Special cases**
   - **(i) General term**
   - **(ii) Middle term**
   - **(iii) Greatest coefficient**
   - **(iv) Coefficient of x^p**
   - **(v) Term dependent of x**
   - **(vi) Greatest term**

Binomial Theorem for positive integral index

Any expression containing two terms only is called binomial expression eg. a+b, 1 + ab etc

For positive integer n

$$(a+b)^n = {}^nc_0 a^n b^0 + {}^nc_1 a^{n-1} b^1 + {}^n c_2 a^{n-2} b^2 + \ldots + {}^nc_{n-1} a^1 b^{n-1} + {}^nc_n a^0 b^n$$

$$= \sum_{r=0}^{n} {}^nc_r a^{n-r} b^r \text{ Binomial theorem}$$

where ${}^nc_r = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!r!} = {}^nc_{n-r}$ for $0 \le r \le n$

are called binomial coefficients.

$${}^nC_r = \frac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots r}$$, numerator contains r factors

$${}^{10}C_7 = \frac{10!}{7!\ 3!} = \frac{10.9.8}{3.2.1} = 120 = {}^{10}C_{10-7} = {}^{10}C_3$$

Pascal's Triangle

1 $\quad (a+b)^0 = 1$

1 1 $\quad (a+b)^1 = 1a + 1b$

1 2 1 $\quad (a+b)^2 = 1a^2 + 2ab + 1b^2$

1 3 3 1 $\quad (a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$

1 4 6 4 1 $\quad (a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

1 5 10 10 5 1 $\quad (a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$

$\Downarrow$

${}^0C_0$

${}^1C_0 \quad {}^1C_1$

${}^2C_0 \quad {}^2C_1 \quad {}^2C_2$

${}^3C_0 \quad {}^3C_1 \quad {}^3C_2 \quad {}^3C_3$

${}^4C_0 \quad {}^4C_1 \quad {}^4C_2 \quad {}^4C_3 \quad {}^4C_4$

${}^5C_0 \quad {}^5C_1 \quad {}^5C_2 \quad {}^5C_3 \quad {}^5C_4 \quad {}^5C_5$

$$\Rightarrow \quad {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$$

Observations from binomial theorem

1. $(a+b)^n$ has n+1 terms as $0 \le r \le n$
2. Sum of indeces of a and b of each term in above expansion is n
3. Coefficients of terms equidistant from beginning and end is same as ${}^nc_r = {}^nc_{n-r}$

$$\boxed{(a+b)^n = {}^nc_0 a^n b^0 + {}^nc_1 a^{n-1} b^1 + {}^nc_2 a^{n-2} b^2 + \dots + {}^nc_{n-1} a^1 b^{n-1} + {}^nc_n a^0 b^n}$$

Special cases of binomial theorem

$$(x-y)^n = {}^nc_0 x^n - {}^nc_1 x^{n-1} y + {}^nc_2\, x^{n-2}\, y^2 \ldots + (-1)^n\ {}^nc_n\, y^n$$

$$= \sum_{r=0}^{n} (-1)^r\ {}^nc_r\, x^{n-r}\, y^r$$

$$(1+x)^n = {}^nc_0 + {}^nc_1 x + {}^nc_2\, x^2 + \ldots + {}^nc_n x^n = \sum_{r=0}^{n} {}^nc_r\, x^r$$

in ascending powers of x

$$(1+x)^n = {}^nc_0 x^n + {}^nc_1 x^{n-1} + \ldots + {}^nc_n = \sum_{r=0}^{n} {}^nc_r\, x^{n-r} = (x+1)^n$$

in descending powers of x

Illustrative Example

Expand $(x + y)^4 + (x - y)^4$ and hence find the value of $\left(\sqrt{2}+1\right)^4+\left(\sqrt{2}-1\right)^4$

Solution :

$$(x+y)^4 = {}^4C_0x^4y^0 + {}^4C_1x^3y^1 + {}^4C_2x^2y^2 + {}^4C_3x^1y^3 + {}^4C_4x^0y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Similarly $(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

$$\therefore (x+y)^4 + (x-y)^4 = 2\left(x^4 + 6x^2y^2 + y^4\right)$$

Hence $\left(\sqrt{2}+1\right)^4+\left(\sqrt{2}-1\right)^4 = 2\left(\sqrt{2}^4 + 6\sqrt{2}^2 1^2 + 1^4\right)$

$$= 34$$

General term of $(a + b)^n$

$$T_{r+1} = {}^nc_r\, a^{n-r}\, b^r, r = 0, 1, 2, \ldots, n$$

$$r = 0, \text{ First Term } T_1 = {}^nc_0 a^n b^0$$

$$r = 1, \text{ Second Term } T_2 = {}^nc_1 a^{n-1} b^1$$

$$T_r = {}^nc_{r-1}\, a^{n-r+1}\, b^{r-1}, r = 1, 2, 3, \ldots, n$$

$$r = 0 \quad 1 \quad 2 \quad 3 \quad 4 \qquad n-1 \quad n$$

$$T_1 \; T_2 \; T_3 \; T_4 \; T_5 \qquad T_n \qquad T_{n+1}$$

n+1 terms

kth term from end is (n-k+2)th term from beginning

Illustrative Example

Find the 6th term in the expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$ and its 4th term from the end.

Solution :

$$T_{r+1} = {}^9C_r\left(\frac{4x}{5}\right)^{9-r}\left(\frac{-5}{2x}\right)^r$$

$$T_6 = T_{5+1} = {}^9C_5\left(\frac{4x}{5}\right)^4\left(\frac{-5}{2x}\right)^5 = -\frac{9!}{4!5!}\frac{4^4 5^5}{5^4 2^5 x}$$

$$= -\frac{9.8.7.6}{4.3.2.1}\frac{2^3.5}{x} \qquad = -\frac{5040}{x}$$

Illustrative Example

Find the 6th term in the expansion of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$ and its 4th term from the end.

Solution :

$$T_{r+1} = {}^9C_r \left(\frac{4x}{5}\right)^{9-r} \left(\frac{-5}{2x}\right)^r$$

4th term from end = 9-4+2 = 7th term from beginning i.e. T_7

$$T_7 = T_{6+1} = {}^9C_6 \left(\frac{4x}{5}\right)^3 \left(\frac{-5}{2x}\right)^6 = \frac{9!}{3!6!} \frac{4^3 5^6}{5^3 2^6 x^3} = \frac{9.8.7}{3.2.1} \frac{5^3}{x^3}$$

$$= \frac{10500}{x^3}$$

Middle term

CaseI: n is even, i.e. number of terms odd only one middle term

$$\left(\frac{n+2}{2}\right)^{th} \text{ term} \qquad T_{\frac{n+2}{2}} = T_{\frac{n}{2}+1} = {}^{n}c_{\frac{n}{2}}\, a^{\frac{n}{2}}\, b^{\frac{n}{2}}$$

CaseII: n is odd, i.e. number of terms even, two middle terms

$$\left(\frac{n+1}{2}\right)^{th} \text{ term} \qquad T_{\frac{n+1}{2}} = T_{\frac{n-1}{2}+1} = {}^{n}c_{\frac{n-1}{2}}\, a^{\frac{n+1}{2}}\, b^{\frac{n-1}{2}}$$

$$\left(\frac{n+3}{2}\right)^{th} \text{ term} \qquad T_{\frac{n+3}{2}} = T_{\frac{n+1}{2}+1} = {}^{n}c_{\frac{n+1}{2}}\, a^{\frac{n-1}{2}}\, b^{\frac{n+1}{2}}$$

Middle term

$$\left(x+\frac{1}{x}\right)^{2n} = ?$$

Greatest Coefficient ${}^{n}c_r,\ 0 \le r \le n$

CaseI: n even

$$\text{Coefficient of middle term } T_{\frac{n}{2}+1} \text{ is max i.e. for } r = \frac{n}{2}$$

$$ {}^{n}C_{\frac{n}{2}} $$

CaseII: n odd

$$\text{Coefficient of middle term } T_{\frac{n+1}{2}} \text{ or } T_{\frac{n+3}{2}} \text{ is max i.e. for } r = \frac{n-1}{2} \text{ or } \frac{n+1}{2}$$

$$ {}^{n}C_{\frac{n-1}{2}} \quad \text{or} \quad {}^{n}C_{\frac{n+1}{2}} $$

Illustrative Example

Find the middle term(s) in the expansion of $\left(3x - \frac{x^3}{6}\right)^7$ and hence find greatest coefficient in the expansion

Solution :

Number of terms is 7 + 1 = 8 hence 2 middle terms, (7+1)/2 = 4th and (7+3)/2 = 5th

$$T_4 = T_{3+1} = {}^7C_3 (3x)^4 \left(\frac{-x^3}{6}\right)^3 = -\frac{7!}{4!3!}\frac{3^4 x^{13}}{6^3}$$

$$= -\frac{7.6.5}{3.2.1}\frac{3x^{13}}{2^3} = -\frac{105}{8}x^{13}$$

Illustrative Example

Find the middle term(s) in the expansion of $\left(3x - \frac{x^3}{6}\right)^7$ and hence find greatest coefficient in the expansion

Solution :

$$T_5 = T_{4+1} = {}^7C_4 (3x)^3 \left(\frac{-x^3}{6}\right)^4 = \frac{7!}{3!4!}\frac{3^3 x^{15}}{6^4}$$

$$= \frac{7.6.5}{3.2.1}\frac{x^{15}}{2^4 3} = \frac{35}{48}x^{15}$$

Hence Greatest coefficient is

$${}^7C_4 \text{ or } {}^7C_3 \text{ or } \frac{7!}{3!4!} = \frac{7.6.5}{3.2.1} = 35$$

Coefficient of x^p in the expansion of $(f(x) + g(x))^n$

Algorithm

Step1: Write general term T_{r+1}

Step2: Simplify i.e. separate powers of x from coefficient and constants and equate final power of x to p

Step3: Find the value of r

Term independent of x in $(f(x) + g(x))^n$

Algorithm

Step1: Write general term T_{r+1}

Step2: Simplify i.e. separate powers of x from coefficient and constants and equate final power of x to 0

Step3: Find the value of r

Illustrative Example

Find the coefficient of x^5 in the expansion of $\left(3x^2 - \frac{1}{2x^3}\right)^{10}$ and term independent of x

Solution :

$$T_{r+1} = {}^{10}C_r \left(3x^2\right)^{10-r} \left(-\frac{1}{2x^3}\right)^r$$

$$= {}^{10}C_r\, 3^{10-r} \left(-\frac{1}{2}\right)^r x^{20-2r-3r}$$

For coefficient of x^5, $20 - 5r = 5 \Rightarrow r = 3$

$$T_{3+1} = {}^{10}C_3\, 3^{10-3} \left(-\frac{1}{2}\right)^3 x^5$$

Coefficient of x^5 = -32805

Solution Cont.

$$T_{r+1} = {}^{10}C_r \left(3x^2\right)^{10-r} \left(-\frac{1}{2x^3}\right)^r$$

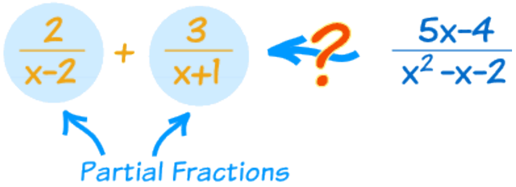
$$= {}^{10}C_r \, 3^{10-r} \left(-\frac{1}{2}\right)^r x^{20-2r-3r}$$

For term independent of x i.e. coefficient of x^0, 20 - 5r = 0 $\Rightarrow$ r = 4

$$T_{4+1} = {}^{10}C_4 \, 3^{10-4} \left(-\frac{1}{2}\right)^4$$

Term independent of x $= \dfrac{76545}{8}$

Partial Fractions



Introduction

- In this chapter you will learn to add fractions with different denominators (a recap)

- You will learn to work backwards and split an algebraic fraction into components called 'Partial Fractions'

Teachings for Exercise 1A

$\frac{2}{x-2} + \frac{3}{x+1} \leftarrow ? \ \frac{5x-4}{x^2-x-2}$

Partial Fractions

Partial Fractions

You can add and subtract several fractions as long as they share a common denominator

You will have seen this plenty of times already! If you want to combine fractions you must make the denominators equivalent…

Calculate:

$$\frac{2}{x+3} - \frac{1}{x+1}$$

$\times \frac{x+1}{x+1}$ $\qquad$ $\times \frac{x+3}{x+3}$

$$\frac{2(x+1)}{(x+3)(x+1)} - \frac{1(x+3)}{(x+3)(x+1)}$$

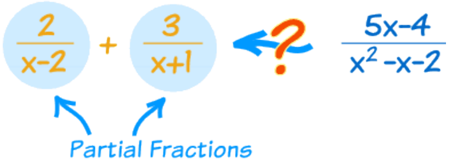
Multiply brackets

$$\frac{2x+2}{(x+3)(x+1)} - \frac{x+3}{(x+3)(x+1)}$$

Group terms

$$= \frac{x-1}{(x+3)(x+1)}$$

Teachings for Exercise 1B



Partial Fractions

You can split a fraction with two linear factors into Partial Fractions

For example: $$\frac{x-1}{(x+3)(x+1)} = \frac{2}{x+3} - \frac{1}{x+1}$$ when split up into Partial Fractions

$$\frac{11}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$ when split up into Partial Fractions

You need to be able to calculate the values of A and B…

$\frac{2}{x-2} + \frac{3}{x+1} \leftarrow ? \quad \frac{5x-4}{x^2-x-2}$

Partial Fractions

Partial Fractions

You can split a fraction with two linear factors into Partial Fractions

Split

$$\frac{6x-2}{(x-3)(x+1)}$$

into Partial Fractions

$$\boxed{\frac{6x-2}{(x-3)(x+1)}}$$

Split the Fraction into its 2 linear parts, with numerators A and B

$$\boxed{\frac{A}{(x-3)} + \frac{B}{(x+1)}}$$

Cross-multiply to make the denominators the same

$$\frac{A(x+1)}{(x-3)(x+1)} + \frac{B(x-3)}{(x-3)(x+1)}$$

Group together as one fraction

$$= \boxed{\frac{A(x+1)+B(x-3)}{(x-3)(x+1)}}$$

This has the same denominator as the initial fraction, so the numerators must be the same

$$6x-2 = A(x+1)+B(x-3)$$

If x = -1:

$$-8 = -4B$$

$$2 = B$$

If x = 3:

$$16 = 4A$$

$$4 = A$$

You now have the values of A and B and can write the answer as Partial Fractions

$$= \boxed{\frac{4}{(x-3)} + \frac{2}{(x+1)}}$$

1B

Teachings for Exercise 1C

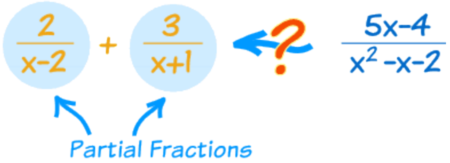
Partial Fractions

You can also split fractions with more than 2 linear factors in the denominator

For example: $$\frac{4}{(x+1)(x-3)(x+4)} = \square + \frac{B}{x-3} + \frac{C}{x+4}$$

when split up into Partial Fractions

1C



Partial Fractions

You can also split fractions with more than 2 linear factors in the denominator

Split

$$\frac{6x^2+5x-2}{x(x-1)(2x+1)}$$

into Partial fractions

$$\boxed{\frac{6x^2+5x-2}{x(x-1)(2x+1)}}$$

Split the Fraction into its 3 linear parts

$$+\frac{B}{x-1}+\frac{C}{2x+1}$$

Cross Multiply to make the denominators equal

$$\frac{A(x-1)(2x+1)}{x(x-1)(2x+1)}+\frac{B(x)(2x+1)}{x(x-1)(2x+1)}+\frac{C(x)(x-1)}{x(x-1)(2x+1)}$$

Put the fractions together

$$\boxed{\frac{A(x-1)(2x+1)+B(x)(2x+1)+C(x)(x-1)}{x(x-1)(2x+1)}}$$

The numerators must be equal

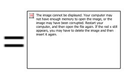
$$6x^2+5x-2 = A(x-1)(2x+1)+B(x)(2x+1)+C(x)(x-1)$$

If x = 1

$$9 = 3B$$

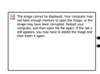
$$3 = B$$

If x = 0

$$-2 =$$ 

$$2 = A$$

If x = -0.5

 $$= 0.75C$$

$$-4 = C$$

You can now fill in the numerators

$$= \boxed{\frac{2}{x}+\frac{3}{x-1}-\frac{4}{2x+1}}$$

Partial Fractions

You can also split fractions with more than 2 linear factors in the denominator

Split

$$\frac{4x^2 - 21x + 11}{x^3 - 4x^2 + x + 6}$$

into Partial fractions

You will need to factorise the denominator first…

$$x^3 - 4x^2 + x + 6$$

$$(1)^3 - 4(1)^2 + (1) + 6 = 4$$

$$(-1)^3 - 4(-1)^2 + (-1) + 6 = 0$$

Try substituting factors to make the expression 0

Therefore (x + 1) is a factor…

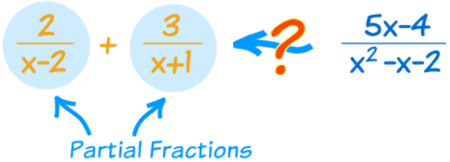
Divide the expression by (x + 1)

$$\begin{array}{r|l}
 & x^2 - 5x + 6 \\
\hline
x + 1 & x^3 - 4x^2 + x + 6 \\
 & x^3 + x^2 \\
\hline
 & -5x^2 + x + 6 \\
 & -5x^2 - 5x \\
\hline
 & 6x + 6 \\
 & 6x + 6 \\
\hline
 & 0
\end{array}$$

$$x^3 - 4x^2 + x + 6 = (x + 1)(x^2 - 5x + 6)$$

$$x^3 - 4x^2 + x + 6 = (x + 1)(x - 2)(x - 3)$$

You can now factorise the quadratic part



Partial Fractions

You can also split fractions with more than 2 linear factors in the denominator

Split

$$\frac{4x^2-21x+11}{x^3-4x^2+x+6}$$

into Partial fractions

$$\frac{4x^2-21x+11}{x^3-4x^2+x+6} = \boxed{\frac{4x^2-21x+11}{(x+1)(x-2)(x-3)}}$$

Split the fraction into its 3 linear parts

$$\boxed{\frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x-3}}$$

Cross multiply

$$\frac{A(x-2)(x-3)}{(x+1)(x-2)(x-3)} + \frac{B(x+1)(x-3)}{(x+1)(x-2)(x-3)} + \frac{C(x+1)(x-2)}{(x+1)(x-2)(x-3)}$$

Group the fractions

The numerators must be equal

$$4x^2-21x+11 = A(x-2)(x-3) + B(x+1)(x-3) + C(x+1)(x-2)$$

If x = 2

$$-15 = -3B$$

$$5 \quad B$$

If x = 3

$$16 =$$

$$-4 = C$$

If x = -1

$$36 = 12A$$

$$3 = A$$

Replace A, B and C

$$= \boxed{\frac{3}{x+1} + \frac{5}{x-2} - \frac{4}{x-3}}$$

Teachings for Exercise 1D

Partial Fractions

$$\begin{array}{r|l} & x \\ \hline x^3+3x & x^4 \qquad +9 \\ & x^4+3x^2 \\ \hline & -3x^2+9 \end{array}$$

*STOP when degree of *remainder* < degree of *divisor*!

*because you can't divide anymore!

You need to be able to split a fraction that has repeated linear roots into a Partial Fraction

For example: $\dfrac{3x^2-4x+2}{(x+1)(x-5)^2} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-5)} + \dfrac{C}{(x-5)^2}$ when split up into Partial Fractions

The repeated root is included once 'fully' and once 'broken down'

1D

Partial Fractions

$$\begin{array}{r} x \\ x^3+3x \overline{\big)\, x^4 \qquad\qquad +9} \\ x^4+3x^2 \qquad \\ \hline -3x^2+9 \end{array}$$

*STOP when degree of _remainder_ < degree of _divisor_!

*because you can't divide anymore!

You need to be able to split a fraction that has repeated linear roots into a Partial Fraction

Split

$$\frac{11x^2+14x+5}{(x+1)^2(2x+1)}$$

into Partial fractions

$$\boxed{\frac{11x^2+14x+5}{(x+1)^2(2x+1)}}$$

Split the fraction into its 3 parts

$$\boxed{\frac{A}{(x+1)}+\frac{B}{(x+1)^2}+\frac{C}{(2x+1)}}$$

Make the denominators equivalent

$$\frac{A(x+1)(2x+1)}{(x+1)^2(2x+1)}+\frac{B(2x+1)}{(x+1)^2(2x+1)}+$$

Group up

$$=\boxed{\frac{A(x+1)(2x+1)+B(2x+1)+C(x+1)^2}{(x+1)^2(2x+1)}}$$

The numerators will be the same

$$11x^2+14x+5 = A(x+1)(2x+1)+B(2x+1)+C(x+1)^2$$

If x = -1

$$2 = -B$$
$$-2 =$$

If x = -0.5

$$0.75 = 0.25C$$
$$3 \quad C$$

At this point there is no way to cancel B and C to leave A by substituting a value in

Choose any value for x (that hasn't been used yet), and use the values you know for B and C to leave A

If x = 0

$$= 1A + 1B \quad 1C$$
$$5 = \quad - 2 + 3$$
$$4 \quad A$$

Sub in the values of A, B and C

$$=\boxed{\frac{4}{(x+1)}-\frac{2}{(x+1)^2}+\frac{3}{(2x+1)}}$$

1D

Teachings for Exercise 1E

$$\begin{array}{r|l} & \qquad\quad x \\ \hline x^3+3x & x^4 \qquad\quad +9 \\ & x^4+3x^2 \\ \hline & \quad -3x^2+9 \end{array}$$

*STOP when degree of *remainder* $<$ degree of *divisor*!

*because you can't divide anymore!

Partial Fractions

You can split an improper fraction into Partial Fractions. You will need to divide the numerator by the denominator first to find the 'whole' part

$$\frac{22}{35} = \frac{1}{5} + \frac{3}{7}$$

A regular fraction being split into 2 'components'

$$\frac{57}{20} = 2 \quad \frac{1}{4} + \frac{3}{5}$$

A top heavy (improper) fraction will have a 'whole number part before the fractions

$$\begin{array}{r}
x \\
x^3+3x \overline{\big) x^4 \qquad +9} \\
x^4 + 3x^2 \\
\hline
-3x^2 + 9
\end{array}$$

*STOP when degree of remainder < degree of divisor!

*because you can't divide anymore!

Partial Fractions

You can split an improper fraction into Partial Fractions. You will need to divide the numerator by the denominator first to find the 'whole' part

Split

$$\frac{3x^2-3x-2}{(x-1)(x-2)}$$

into Partial fractions

Remember, Algebraically an 'improper' fraction is one where the degree (power) of the numerator is equal to or exceeds that of the denominator

$$\frac{3x^2-3x-2}{(x-1)(x-2)} = \frac{3x^2-3x-2}{x^2-3x+2}$$

Divide the numerator by the denominator to find the 'whole' part

$$\begin{array}{r|l}
 & 3 \\
x^2-3x+2 & 3x^2-3x-2 \\
 & 3x^2-9x+6 \\
\hline
 & 6x-8
\end{array}$$

Now rewrite the original fraction with the whole part taken out

$$\frac{3x^2-3x-2}{(x-1)(x-2)} = 3 + \boxed{\frac{6x-8}{(x-1)(x-2)}}$$

Split the fraction into 2 parts (ignore the whole part for now)

$$\boxed{\frac{A}{(x-1)} + \frac{B}{(x-2)}}$$

Make denominators equivalent and group up

$$= \boxed{\frac{A(x-2)+B(x-1)}{(x-1)(x-2)}}$$

The numerators will be the same

$$6x-8 = A(x-2)+B(x-1)$$

If x = 2 $\quad 4 = B$

If x = 1 $\quad -2 = -A$

$$2 \quad A$$

$$= \boxed{3 + \frac{2}{(x-1)} + }$$

1E

Summary

- We have learnt how to split Algebraic Fractions into 'Partial fractions'

- We have also seen how to do this when there are more than 2 components, when one is repeated and when the fraction is 'improper'