HARYANA STATE BOARD OF TECHNICAL EDUCATION

LEARNING TEXT BOOKLET (MATHEMATICS) DIPLOMA 1ST YEAR (January, 2019)

Developed By

Haryana State Board of Technical Education, Bays 7-12, Sector 4, Panchkula

In collaboration with

National Institute of Technical Teachers Training & Research, Sector-26, Chandigarh

PREFACE

Technical Education in polytechnics plays a very vital role in human resource development of the country by creating skilled manpower, enhancing industrial productivity and improving the quality of life. The aim of the polytechnic education in particular is to create a pool of skill based manpower to support shop floor and field operations as a bridge between technician and engineers. Moreover, a small and medium scale industry prefers to employ diploma holders because of their special skills in reading and interpreting drawings, estimating, costing and billing, supervision, measurement, testing, repair, maintenance etc.

Despite the plethora of opportunities available for the diploma pass-out students, the unprecedented expansion of the technical education sector in recent years has brought in its wake questions about the quality of education imparted. Moreover, during the last few years the students seeking admissions in the polytechnics are coming mainly from the rural background and face the major challenge of learning and understanding the technical contents of various subjects in English Language.

The major challenge before the Haryana State Board of Technical Education is to ensure the quality of a technical education to the stakeholders along its expansion. In order to meet the challenges and requirement of future technical education manpower, consistent efforts are made by Haryana State Board of Technical Education to design need based diploma programmes in collaboration with National Institute of Technical Teachers Training and Research, Chandigarh as per the new employment opportunities.

The Board undertook the development of the learning material tailored to match the curriculum content. This learning Text Booklet shall provide a standard material to the teachers and students to aid their learning and achieving their study goals.

Secretary
HSBTE, Panchkula

ACKNOWLEDGEMENT

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Joint Secretary
HSBTE, Panchkula

TEAM INVOLVED IN DEVELOPMENT OF LEARNING TEXTBOOK (MATHEMATICS)

ADMINISTRATIVE AUTHORITIES

1. Sh. Anil Kumar, IAS, Additional Chief Secretary, Technical Education, Govt. of Haryana-Cum-Chairman, Haryana State Board of Technical Education, Panchkula
2. Sh. A. Sreeniwas, IAS, Director General, Technical Education, Govt. of Haryana
3. Sh. K.K. Kataria, Director, Technical Education-Cum-Secretary, Haryana State Board of Technical Education, Panchkula
4. Dr. S.S. Pattnaik, Director, National Institute of Technical Teachers Training and Research, Chandigarh

POLYTECHNIC FACULTY

5. Sh. R.D. Sharma, Lecturer Mathematics, Govt. Polytechnic, Bhiwani
6. Mrs. Urmil Chaudhary, Lecturer Mathematics, Govt. Polytechnic, Sirsa
7. Sh. Satyavan Dhaka, Lecturer Mathematics, Govt. Polytechnic, Nilokheri
8. Mrs. Sapna Sang, Lecturer Mathematics, SJPP, Damla
9. Sh. Ravi Bansal, Lecturer Mathematics, Govt. Polytechnic, Manesar

ACADEMIC EXPERTS

10. Dr. K.C. Lachhwani, Assistant Professor, Applied Maths, Applied Science Deptt. NITTTR, Chandigarh
11. Dr. K.K. Gogna, Associate Professor, Applied Maths, Applied Science Deptt., Punjab Engineering College, Chandigarh
12. Dr. Kalpana Dahiya, Assistant Professor, Applied Maths, Applied Science Deptt. UIET, Panjab University, Chandigarh

COORDINATORS

13. Dr. B.C. Choudhary, Professor, Applied Physics, Applied Science Dept., NITTTR, Chandigarh
14. Sh. R.K. Miglani, Joint Secretary (Academics), Haryana State Board of Technical Education, Panchkula
15. Dr. Nidhi Aggarwal, Assistant Secretary, Haryana State Board of Technical Education, Panchkula
16. Sh. Sanjeev Kumar, Assistant Secretary, Haryana State Board of Technical Education, Panchkula
17. Mrs. Geeta Gulia, Principal, Govt. Poly. Jhajjar

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SYLLABUS

1.2 APPLIED MATHEMATICS

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Section-A **(20%)**

DETAILED CONTENTS

1. Algebra **(30 Hrs)**

- Law of Indices, Formula of Factorisation and expansion i.e. $(a+b)^2$, (a^3+b^3) etc.
- Partial fraction:- Definition of Polynomial fraction proper & improper fractions and definition of partial fractions. To resolve proper fraction into partial fraction with denominator containing non-repeated linear factors, only.
- Complex numbers: definition of complex number, real and imaginary parts of a complex number, Polar and Cartesian Form and their inter conversion, Conjugate of a complex number, modulus and amplitude, addition subtraction, multiplication and division of complex number.
- Determinants and Matrices – Evaluation of determinants (up to 3 order) by laplace method. Solution of equations (up to 3 unknowns) by Cramer's Rule. Definition of Matrices and types, addition subtraction and multiplication of Matrices (up to 2 order).
- Permutation, combination formula, Values of ${}^nP_r$ and ${}^nC_r$.
- Binomial theorem for positive integral index , General term, simple problems

Section –B **(20%)**

2. Trigonometry **(14 Hrs)**

- Concept of angle: measurement of angle in degrees, grades, radians and their conversions.
- T-Ratios of standard angle (0^0,30^0,45^0etc) and fundamental Identities, Allied angles (without proof) Sum, Difference formulae and their applications (without proof). Product formulae (Transformation of product to sum, difference and vice versa)
- Applications of Trigonometric terms in engineering problems such as to find an angle of elevation, height, distance etc.

3. Co-ordinate Geometry **(12 Hrs)**

- Point: Distance Formula, Mid Point Formula, Centroid of triangle and area of triangle.

- Straight line: Slope of a line, equation of straight line in various standards forms (without proof); (slope intercept form, intercept form, one-point form, two-point form, normal form, general form), angle between two straight lines.

- Circle: General equation of a circle and identification of centre and radius of circle. To find the equation of a circle, given:
  - * Centre and radius
  - * Coordinates of end points of a diameter

Section –C **(60%)**

4. Differential Calculus **(40 Hrs)**

- Definition of function; Concept of limits (Introduction only) and problems related to four standard limits only.

- Differentiation of standard function (Only formulas), Differentiation of Algebraic function, Trigonometric functions, Exponential function, Logarithmic function

- Differentiation of sum, product and quotient of functions.

- Successive differentiation (up to 2nd order)

- Application of differential calculus in:
  - (a) Rate measures
  - (b) Maxima and minima

5. Integral Calculus **(28 Hrs)**

- Integration as inverse operation of differentiation with simple exs.

- Simple standard integrals, Integrations by parts and related Simple problems

- Evaluation of definite integrals with given limits.

  Evaluation of $\int_0^{\frac{\pi}{2}} sin^n x\,dx$, $\int_0^{\frac{\pi}{2}} cos^n x\,dx$, $\int_0^{\frac{\pi}{2}} sin^m x\,cos^n x\,dx$
  using formulae without proof (m and n being positive integers only) using pre-existing mathematical models.
- Applications of integration: for evaluation of area under a curve and axes (Simple problems where the limits are given).

- Numerical integration by Trapezoidal Rule and Simpson's 1/3rd Rule using pre-existing mathematical models

6. Differential Equations
(04 Hrs)

Definition, order, degree and linearity, of an ordinary differential equation. Solution of Ist order and Ist degree differential equation by variable separable method (Simple problems)

7. Statistics **(12 Hrs)**

- Measures of Central Tendency: Mean, Median, Mode
- Measures of Dispersion: Mean deviation from mean, Standard deviation
- Correlation coefficient and Coefficient of rank correlation (Simple problems)

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DISTRIBUTION OF SYLLABUS FOR ASSESSMENTS & DISTRIBUTION OF MARKS

Section	**Assessment**	**Units to be covered**		**Distribution of Marks**
A	1st Internal	Unit 1:	Algebra	20
B	2nd Internal	Unit 2:	Trigonometry	10
		Unit 3:	Coordinate Geometry	10
C	Final	Unit 4	Differential Calculas	20
		Unit 5	Integral Calculas	16
		Unit 6	Differential equations	8
		Unit 7	Statistics	16

UNIT– 1

ALGEBRA

Learning objectives

- To understand and identify the basic features of law of indices, formulae of algebra, partial fractions, complex numbers, determinants & matrices, permutation & combination, binomial theorem in positive integral index.

1.1 LAW OF INDICES

Introduction: A power or an index is used to write product of numbers very compactly. The plural of index is indices. In this topic, we remind you how this is done and state a number of rules or laws, which can be used to simplify expressions involving indices.

Power or Indices : We write the expression $3 \times 3 \times 3 \times 3$ as 3^4.

We read this as "three to the power four or three raise to power four".

In the expression b^c, b is called the base and c is called the index.

Rules or Laws of Indices	Examples
First Rule $a^m \times a^n = a^{m+n}$	$2^5 \times 2^3 = 2^8$
(ii) $\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{5^7}{5^3} = 5^{7-3} = 5^4$
(iii) $(a^m)^n = a^{mn}$	$(10^3)^7 = 10^{21}$
(iv) $a^0 = 1$	$10^0 = 1$
(v) $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$	$\left(\dfrac{5}{6}\right)^2 = \dfrac{25}{36}$
(vi) $(ab)^n = a^n b^n$	$(2a)^5 = 2^5a^5 = 32a^5$
(vii) $a^{-m} = \dfrac{1}{a^m}$	$(9)^{-2} = \frac{1}{9^2} = \frac{1}{81}$

(viii) $a^{\frac{n}{m}} = \sqrt[m]{a^n}$ $\qquad 8^{2/3} = \sqrt[3]{8^2} = (8)^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^2 = 4$

(ix) $a^m = b^m \Rightarrow a = b$ $\qquad a^5 = b^5 \Rightarrow a = b$

(if the powers are equal then bases are equal)

(x) If $a^m = a^n \Rightarrow m = n$

(xi) $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab} = (ab)^{\frac{1}{n}}$

Examples:

(i) $a^2 \times a^5 = a^7$

(ii) $a^{-2}\, b^3 \times a^5\, b^{-4} = a^3\, b^{-1}$

(iii) $\left(\frac{3a^{-2}}{b^{-1}}\right)^3 = \frac{3^3 a^{-6}}{b^{-3}} = \frac{27b^3}{a^6}$

(iv) $\left(\frac{5a^2}{3b^3}\right)^2 \times \left(\frac{a}{b}\right)^{-2} = \frac{25a^4}{9b^6} \times \frac{a^{-2}}{b^{-2}} = \frac{25a^2}{9b^4}$

(v) $3^3 (x^3)^3 \times (y^4)^3 = 27x^9 y^{12}$

(vi) $\frac{a \times (ab^4)^2}{(a^2 \times b)^3} = \frac{a \times a^2 b^8}{a^6 b^3} = \frac{a^3 b^8}{a^6 b^3} = a^{-3} b^5$

(vii) $\sqrt{\frac{a^2}{b^6}} = (a^2 b^{-6})^{\frac{1}{2}} = (a^2)^{\frac{1}{2}} \times (b^{-6})^{\frac{1}{2}} = a \times b^{-3} = \frac{a}{b^3}$

EXERCISE - I

1. Simplify the following:

(i) $2^3 \times 2^4$

(ii) $8^{13} \div 8^5$

(iii) $(a^3)^4$

(iv) $3a^2 b^3 \times 4a^4 b^5$

(v) $\dfrac{a^2}{b^{-3}c^2}$

(vi) $\left(\dfrac{3a^{-2}}{b^{-1}}\right)^3$

(vii) $\left(\dfrac{5a^2}{3b^3}\right)^2 \times \left(\dfrac{a}{b}\right)^{-2}$

(viii) $\dfrac{a^3 \times a^4}{a^2}$

(ix) 7^{-2}

(x) $(3x^3 y^4)^3$

ANSWERS

(i) 2^7 (ii) 8^8 (iii) a^{12} (iv) $12a^6 b^8$ (v) $a^2 b^3 c^{-2}$ (vi) $\dfrac{27b^3}{a^6}$

(vii) $\dfrac{25a^2}{9b^4}$ (viii) a^5 (ix) $\dfrac{1}{49}$ (x) $27x^9 y^{12}$

Formulae of Algebra

For any two numbers a and b

(1) $(a + b)^2 = a^2 + 2ab + b^2$ $\Rightarrow$ Square of a sum

(2) $(a - b)^2 = a^2 - 2ab + b^2$ $\Rightarrow$ Square of a difference

(3) $a^2 - b^2 = (a + b)(a - b)$ $\Rightarrow$ Difference of two squares

(4) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ $\Rightarrow$ Difference of two cubes

(5) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ $\Rightarrow$ Sum of two cubes

(6) $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$ $\Rightarrow$ Cube of a sum

(7) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$ $\Rightarrow$ Cube of a difference

Factorization Formula/Quadratic Formula

If a, b and c are real numbers, then $ax^2 + bx + c = 0$ has solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: A quadratic equation can be solved by

(1) Using the concept of factorization

(2) Using the concept of quadratic formula

Example 1. Solve by factorization method; $x^2 + 7x + 10 = 0$.

Sol. In this method split the middle part 7 into two parts, such that their sum is +7 and product is +10. So numbers are 2 and 5.

$x^2 + (2x + 5x) + 10 = 0$ $\Rightarrow$ $1.x^2 + 7.x + 10 = 0$

$x^2 + 2x + 5x + 10 = 0$ $\Rightarrow$ $x(x + 2) + 5(x + 2) = 0$

$(x + 2)(x + 5) = 0$

So either $x + 2 = 0$ or $x + 5 = 0$

If $x + 2 = 0$ $\Rightarrow$ $x = -2$

If $x + 5 = 0$ $\Rightarrow$ $x = -5$

Thus, −2, −5 are roots of given equation.

Example 2. Solve the quadratic equation by quadratic formula: $x^2 + 7x + 10 = 0$.

Sol. Comparing the given quadratic equation with $ax^2 + bx + c = 0$, we get

$a = 1, \; b = 7, \; c = 10$

Applying formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ we get}$$

$$\Rightarrow \qquad x = \frac{-7 \pm \sqrt{(7)^2 - 4(1)(10)}}{2(1)}$$

$$= \frac{-7 \pm \sqrt{49 - 40}}{2} = \frac{-7 \pm \sqrt{9}}{2}$$

$$= \frac{-7 \pm 3}{2} = \frac{-7 + 3}{2}, \frac{-7 - 3}{2}$$

$$= \frac{-4}{2}, \frac{-10}{2} = -2, -5$$

Thus, $-2, -5$ are roots of given equation.

1.2 PARTIAL FRACTION

Fraction : An expression of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$ is known as a fraction.

Polynomial : An expression of the type $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n$, where $a_0, a, \ldots, a_n$ are constants, is called a polynomial.

For example: (a) $x^3 + 2x^2 + 7x + 2$, (b) $4x^4 + 7x^3 - 9x^2 + 3$

Degree of Polynomial : Degree of Polynomial is the power of highest term in x(variable). In example (a), degree is 3 and in example (b) degree is 4.

Polynomial with different degree's

Name	Degree	Exs
Constant	zero	7, 9, $\frac{11}{2}$ etc.
Linear	1	x + 1, x - 3, 5x - 3, etc.
Quadratic	2	$x^2 + 7x + 10$, $3x^2 + 7x - 11$ etc.
Cubic	3	$x^3 + 2x^2 + 7x + 2$, $x^3 - 1$ etc.

Rational Fraction : An expression of the type $\frac{p(x)}{q(x)}$, where p(x) and q(x) are polynomials and $q(x) \neq 0$ is known as rational fraction.

For example; $\frac{2x+5}{x^2-5x+4}, \frac{1}{x^2-1}, \frac{x^2-5}{x^2-3x+2}$

A fraction is of two types :

(1) Proper Fraction : If degree of numerator is lower than degree of denominator, it is called proper fraction.

For example; $\frac{2x+5}{x^2-5x+4}, \frac{2x+1}{(2x-1)(x+2)}$ are proper fractions.

(2) Improper Fraction: If the degree of numerator a greater or equal to the degree of denominator, then it is called improper fraction.

For example; $\frac{x^3-5}{x^2-7x+12}, \frac{x^2-5}{x^2-3x+2}$ are improper fractions.

Partial Fraction: The simplest constituent fraction of a compound fraction is called its partial fraction and the process of separating a compound fraction into its simplest constituent fractions is called the resolution into partial fraction.

We have learnt this in previous class that

$$\frac{1}{x-2} + \frac{1}{x-3} = \frac{2x-5}{x^2-5x+6} \text{ (compound fraction)}$$

or $\quad \dfrac{1}{x-2}+\dfrac{1}{x-3}$ are partial fractions of $\dfrac{2x-5}{x^2-5x+6}$

We shall now study how to perform the inverse process i.e. to decompose or break up a single fraction into a number of fractions having their denominator as the factor of denominator of original fraction.

Note: Improper fraction can be converted into proper by dividing numerator by denominator and written in the form :

Improper fraction = quotient + proper fraction

e.g. $$\frac{x^3}{x^2-3x+2}=(x+3)+\frac{7x-6}{x^2-3x+2}$$

Now $\dfrac{7x-6}{x^2-3x+2}$ is a proper fraction and can be split into partial fractions.

Note : For resolving a improper fraction into partial fractions, first it should be converted into a proper fraction as explained above.

We have different types :

Type 1 : To resolve proper fraction into partial fraction with denominator containing non repeated linear factors only

For the proper fraction $\dfrac{p(x)}{q(x)}$, $q(x) \neq 0$ and degree of p(x) < degree of q(x)

The linear factors (ax + b), (cx + d) etc. split into addition with numerator A, B, etc.

For examples:

(i) $$\frac{1}{(ax+b)(cx+d)}=\frac{A}{ax+b}+\frac{B}{cx+d}$$

(ii) $$\frac{1}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2}$$

(iii) $$\frac{1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

where A, B & C are constants.

Example 3. Resolve $\frac{1}{(x+2)(x-3)}$ into partial fractions.

Sol. Given fraction is in proper form.

Consider $$\frac{1}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}$$

Multiplying by the LCM (x + 2) (x – 3) on both sides, we get

$$1 = A(x-3) + B(x+2)$$

(1)

To find A, Put $x + 2 = 0 \Rightarrow x = -2$ in (1)

$$1 = A(-2-3) + B(0)$$

$$1 = A(-5) + 0 = -5A$$

$$A = -\frac{1}{5}$$

To find B, Put $x - 3 = 0 \Rightarrow x = 3$ in (1)

$$1 = A(0) + B(3+2)$$

$$1 = 0 + 5B = 5B$$

$$B = \frac{1}{5}$$

Substitute those value of A and B in (1)

$$\frac{1}{(x+2)(x-3)} = \frac{-\frac{1}{5}}{x+2} + \frac{\frac{1}{5}}{x-3}$$

$$= -\frac{1}{5(x+2)} + \frac{1}{5(x-3)}$$

Example 4. Resolve $\frac{2x-1}{x^2-8x+15}$ into partial fractions.

Sol. Since $x^2 - 8x + 15 = (x - 5)(x - 3)$

Consider $$\frac{2x-1}{x^2-8x+15} = \frac{2x-1}{(x-5)(x-3)} = \frac{A}{x-5} + \frac{B}{x-3}$$

Multiplying by the LCM $(x - 5)(x - 3)$ on both sides

$$2x - 1 = A(x - 3) + B(x - 5) \qquad (1)$$

To find A, put $x - 5 = 0$ i.e. $x = 5$ in (1)

$$2(5) - 1 = A(5 - 3) + B(0)$$

$$9 = 2A + 0 = 2A \qquad \Rightarrow \qquad A = \frac{9}{2}$$

To find B, put $x - 3 = 0 \qquad \Rightarrow \qquad x = 3$ in (1)

$$2(3) - 1 = A(0) + B(3 - 5)$$

$$5 = B(-2) = -2B$$

$$\Rightarrow \qquad B = -\frac{5}{2}$$

Substituting value of A & B, the equation (1) become

$$\frac{2x-1}{x^2-8x+15} = \frac{\frac{9}{2}}{x-5} + \frac{-\frac{5}{2}}{x-3}$$

$$= \frac{9}{2(x-5)} - \frac{5}{2(x-3)}$$

Example 5. Resolve $\frac{x^3}{x^2-3x+2}$ into partial fractions.

Sol. Here given fraction is not in a proper fraction. Dividing x^3 by $x^2 - 3x + 2$, we get

$$\frac{x^3}{x^2-3x+2} = x+3+\frac{7x-6}{x^2-3x+2}$$

Now $\frac{7x-6}{x^2-3x+2}$ is in proper form and can be split into partial fractions.

Let $$\frac{7x-6}{x^2-3x+2} = \frac{7x-6}{(x-1)(x-2)} = \frac{A}{x-1}+\frac{B}{x-2}$$

i.e. $$7x - 6 = A(x - 2) + B(x - 1) \qquad (1)$$

To find A, put $x - 1 = 0$ i.e. $x = 1$ in (1), we get

$$1 = A(1 - 2) + B(0)$$

$$1 = -A \qquad \Rightarrow \qquad A = -1$$

To find B, Put $x - 2 = 0 \Rightarrow x = 2$ in (1), we get

$$7(2) -6 = A(2 - 2) + B(2-1)$$

$$\Rightarrow \quad 14 - 6 = A(0) + B(1)$$

$$\Rightarrow \quad 8 = 0 + B$$

$$\Rightarrow \quad B = 8$$

Putting value of A and B in (A), we get

$$\frac{7x-6}{x^2-3x+2} = \frac{-1}{x-1}+\frac{8}{x-2}$$

$$\therefore \qquad \frac{x^3}{x^2-3x+2} = x+3+\frac{-1}{x-1}+\frac{8}{x-2}$$

$$= x + 3 - \frac{1}{x-1}+\frac{8}{x-2}$$

Example 6. Resolve $\frac{x-4}{(x+4)(x^2-3x+2)}$ into partial fractions.

Sol. Since $x^2 - 3x + 2 = (x - 2)(x - 1)$

$$\therefore \qquad \frac{x-4}{(x+4)(x^2-3x+2)} = \frac{x-4}{(x+4)(x-2)(x-1)}$$

Let $$\frac{x-4}{(x+4)(x^2-3x+2)} = \frac{A}{x+4} + \frac{B}{x-2} + \frac{C}{x-1} \qquad (1)$$

Multiplying the LCM (x + 4)(x – 2) (x – 1) on both side of (1)

$$x - 4 = A(x - 1)(x - 2) + B(x + 4)(x - 1) + C(x + 4)(x - 2) \qquad (2)$$

To find A, Put $x + 4 = 0 \Rightarrow x = -4$ in (2)

$$-8 = A(-4 - 1)(-4 - 2) = A(-5)(-6)$$

$$-8 = A(30) \quad \Rightarrow \quad A = -\frac{8}{30} = -\frac{4}{15}$$

$$A = -\frac{4}{15}$$

To find B, put $x - 2 = 0 \quad \Rightarrow \quad x = 2$ in (2)

$$-2 = B(2 + 4)(2 - 1)$$

$$-2 = B(6)(1) = 6B$$

$$B = -\frac{2}{6} = -\frac{1}{3} \quad \Rightarrow \quad B = -\frac{1}{3}$$

To find C, put $x - 1 = 0 \Rightarrow x = 1$ in (2)

$$-3 = C(1 + 4)(1 - 2) = C(5)(-1) = -5C$$

$$C = \frac{3}{5}$$

Putting values of A, B & C in eqn**.** (1)

$$\frac{x-4}{(x+4)(x^2-3x+2)} = \frac{-4}{15(x+4)} - \frac{1}{3(x-2)} + \frac{3}{5(x-1)}$$

Type (ii): When the denominator contains repeated linear factors

Type (iii): When the denominator contains non-repeated quadratic factors

Type (iv): When the denominator contains repeated quadratic factors

EXERCISE - II

1. Resolve into the partial fractions :

(i) $\frac{1}{(x-3)(x-5)}$

(ii) $\frac{7x+1}{x^2-x-2}$

(iii) $\frac{5x-1}{(x-2)(x+1)}$

(iv) $\frac{x+1}{(x+3)(x^2-4)}$

(v) $\frac{2x-3}{(x-2)(x+3)}$

(vi) $\frac{1}{(1-x)(1-2x)(1-3x)}$

(vii) $\frac{5x-2}{x^2-2x-8}$

(viii) $\dfrac{x^3}{x^2-3x+2}$

(ix) $\dfrac{x^2}{(x-1)(x-2)(x-3)}$

(x) $\dfrac{x+5}{x^2+x}$

ANSWERS

(i) $\dfrac{-\frac{1}{2}}{x-3}+\dfrac{\frac{1}{2}}{x-5}$ (ii) $\dfrac{2}{x+1}+\dfrac{5}{x-2}$ (iii) $\dfrac{3}{x-2}+\dfrac{2}{x+1}$ (iv) $\dfrac{3}{20(x-2)}+\dfrac{1}{4(x+3)}=\dfrac{2}{5(x+2)}$

(v) $\dfrac{7}{5(x-2)}+\dfrac{7}{5(x+3)}$ (vi) $\dfrac{1}{2(1-x)}-\dfrac{4}{1-2x}+\dfrac{9}{2(1-3x)}$ (vii) $\dfrac{3}{x-4}+\dfrac{2}{x+2}$

(viii) $x+3-\dfrac{1}{x-1}+\dfrac{8}{x-2}$ (ix) $\dfrac{1}{2(x-1)}-\dfrac{4}{x-2}+\dfrac{9}{2(x-3)}$ (x) $\dfrac{5}{x}-\dfrac{4}{x+1}$

1.3 COMPLEX NUMBERS

Number System: We know the number system as

(1) Natural numbers, N = {1, 2, 3, …, }

(2) Whole Numbers, W = {0,1, 2, 3, …}

(3) Integers, Z = {…,–3, –2, –1, 0, 1, 2, 3, …}

(4) Rational numbers, Q = $\left\{\dfrac{p}{q}, p, q \in Z, q \neq 0\right\}$

(5) Irrational numbers – The numbers whose decimal representation is non–terminating and non repeating

e.g. $\sqrt{2}, \sqrt{3}$ etc.

(6) Real numbers (R) = (Rational numbers + Irrational numbers)

Let us take an quadratic equation; $x^2 + 7x + 12 = 0$ which has real root –4 and –3, *i.e.* solution of $x^2 + 7x + 12 = 0$ is $x = -4$ and $x = -3$ which are both real numbers.

But for the quadratic equation of the form $4x^2 - 4x + 5 = 0$, no real value of x satisfies the equation. For the solution of the equations of such types the idea of complex numbers is introduced.

Imaginary Numbers or Complex numbers:

Solution of quadratic equation $x^2 + 4 = 0$

or $x^2 = -4$

$$x = \sqrt{-4} = \sqrt{-1 \times 4} = \sqrt{-1} \times \sqrt{4} = i \times 2$$

where $i = \sqrt{-1}$ is an imaginary number called as iota.

The square root of a negative number is always imaginary number.

e.g. $\sqrt{-4}, \sqrt{-16}, \sqrt{-25}, \sqrt{-\frac{9}{16}}$ etc. are all imaginary numbers.

i.e. $\sqrt{-4} = 2i,\ \sqrt{-16} = 4i,\ \sqrt{-25} = 5i$

and $\sqrt{-\frac{9}{16}} = \frac{3}{4}i$

Thus the solution of quadratic equation $4x^2 - 4x + 5 = 0$ are :

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(5)}}{2 \times 4}$$

$$x = \frac{4 \pm \sqrt{16 - 80}}{8}$$

$$x = \frac{4 \pm \sqrt{-64}}{8} = \frac{4 \pm 8i}{8} = \frac{4 + 8i}{8}, \frac{4 - 8i}{8}$$

$$\Rightarrow \quad \frac{1 + 2i}{2}, \frac{1 - 2i}{2}$$

Thus the given quadratic equation has complex roots.

Powers of iota (i)

(i) $i = \sqrt{-1}$

(ii) $i^2 = 1$

(iii) $i^3 = i^2.i = (-1)i = -i$

(iv) $i^4 = (i^2)^2 = (-1)^2 = 1$

(v) $i^5 = i^4.i = i$

(vi) $i^{24} = (i^4)^6 = (1)^6 = 1$

(vii) $i^{25} = (i^4)^6 i = 1 \times i = i$

(viii) $i^{4n+1} = (i^4)^n\, i = 1 \times i = i$

Real and Imaginary part of Complex number : A number of the form x + iy, where x and y are real numbers and i is an imaginary number with property $i^2 = -1$ i.e. $i = \sqrt{-1}$ is called a complex number. The complex number is denoted by Z.

$$\therefore \qquad Z = x + i\,y$$

Here x is called real part of Z and denoted by Re(Z) and y is called imaginary part of Z, it is denoted by Im(Z).

Thus complex number Z = x + iy can be represented as

Z = real part + ***i*** (Imaginary part)

Examples of complex numbers are :

(i) Z = 2 + i, where Re(Z) = 2, Im(Z) = 1

(ii) Z = 4 – 7i, where Re(Z) = 4, Im(Z) = –7

(iii) $Z = -\frac{2}{3}+\frac{5}{3}i$, where $Re(Z) = -\frac{2}{3}$, $Im(Z) = \frac{5}{3}$

(iv) $Z = 4 + 0i$, where $Re(Z) = 4$, $Im(Z) = 0$

Here $Im(Z) = 0$, such type of complex number is known as purely real

(v) $Z = 0 + 3i$, where $Re(Z) = 0$, $Im(Z) = 3$

Hence $Re(Z) = 0$, hence given complex number is called purely imaginary number.

Properties of Complex Numbers:

(i) Equality of two complex number: Let $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$ are two complex numbers.

If $Z_1 = Z_2 \Rightarrow x_1 + iy_1 = x_2 + iy_2$

Then $x_1 = x_2$ and $y_1 = y_2$

i.e. their real and imaginary part are separately equal.

(ii) if $x + iy = 0$, then $x = 0$ and $y = 0$ *i.e.* if a complex number is zero then its real part and imaginary part both are zero.

Example 7. Solve the equation $2x + (3x + y)i = 4 + 10i$.

Sol. Using property (i)

$2x = 4$ and $3x + y = 10$

$\Rightarrow$ $x = 2$, Putting value of $x = 2$ in

$3x + y = 10$, we get $6 + y = 10 \Rightarrow y = 10 - 6 = 4$

Hence $x = 2$, $y = 4$

Example 8. Find x & y if $\frac{1}{x}+\frac{1}{y}i = 2+3i$.

Sol. Equating real and imaginary part, we get

$$\frac{1}{x} = 2 \qquad \Rightarrow \qquad x = \frac{1}{2}$$

and $$\frac{1}{y} = 3 \qquad \Rightarrow \qquad y = \frac{1}{3}$$

x–axis is called real axis
y–axis is called imaginary.

Conjugate of a Complex Number

If z = x + *i*y is a complex number than conjugate of Z is x – *i*y. It is denoted by $\bar{z}$.

$$\therefore \qquad \bar{z} = x - iy$$

e.g. if z = 2 + 3i, then conjugate of z is $\bar{z} = 2 - 3i$.

Properties of conjugate of complex number

(i) $\overline{(\bar{z})} = z$

(ii) $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$

(iii) $\overline{z_1 z_2} = \bar{z}_1 . \bar{z}_2$

Example 9. Write conjugate of $z = 4i^3 + 3i^2 + 5i$.

Sol. Given that

$$z = 4i^3 + 3i^2 + 5i$$

$$= 4i^2 i + 3(-1) + 5i$$

$$z = -4i - 3 + 5i = -3 + i$$

Conjugate of z is –3 – i.

Sign of complex number in quadrant system (Fig. 1.1)

Figure .1.1

Polar and Cartesian form of a complex number. III

Let z = x + iy is a complex number in Cartesian form represented by a Point P(x, y) in Argand plane (XY−plane) as shown in Fig. 1.2 Y

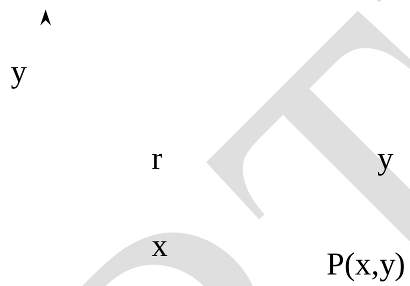

Figure .1.2

Then $\sin\theta = \dfrac{y}{r}$ and $\cos\theta = \dfrac{x}{r}$

$\therefore$ $y = r\sin\theta$ …(1) and $x = r\cos\theta$ (2)

The complex number $Z = x + iy = r\cos\theta + i\, r\sin\theta = r[\cos\theta + i\sin\theta]$ is a polar form of given complex number. Squaring and adding (1) and (2), we get

$$x^2 + y^2 = r^2[\cos^2\theta + \sin^2\theta]$$

$$= r^2[1]$$

$\Rightarrow$ $r = \sqrt{x^2 + y^2}$ $\Rightarrow$ Modulus of z = x + iy *i.e.* r is called the modulus of z

Dividing (1) by (2), we get

$$\tan\theta = \frac{y}{x}$$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow$ The argument or amplitude of z. *i.e.* θ is called the argument or amplitude of z.

$\therefore$ Polar form of a complex number z is

$$Z = x + iy = r[\cos\theta + i\ \sin\theta]$$

Thus Z = x + iy is Cartesian form of Z.

and $Z = r[\cos\theta + i \sin\theta]$ is Polar form of Z

Where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Conversion from Cartesian Form to Polar Form

Let x + iy be a complex number in Cartesian form, then we have to convert into polar form

$$x + iy \xrightarrow{\text{Change}} r(\cos\theta + i\sin\theta)$$

Putting the value of r and θ, we get required form

i.e. $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\therefore$ $x + iy = r(\cos\theta + i\sin\theta)$

$$= \sqrt{x^2 + y^2}\left[\cos(\tan^{-1}\left(\frac{y}{x}\right)\right] + i\sin\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Example 10. Express $1+\sqrt{3}i$ into polar form.

Sol. Let z = $1+\sqrt{3}i$, here x = 1, $y = \sqrt{3}$

Polar form of complex number is

$$z = r[\cos\theta + i \sin\theta]$$

We know, $r = \sqrt{x^2 + y^2} = \sqrt{1+3} = \sqrt{4} = 2$

$$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{1} = \sqrt{3} = \tan\frac{\pi}{3}$$

$\Rightarrow$ $$\theta = \frac{\pi}{3}$$

$\therefore$ Required polar form is

$$z = r[\cos\theta + i\sin\theta] = 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$$

Example 11. Convert 1 – i into polar form.

Sol. Let Z = 1 – i, then x = 1, y = –1

$$r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1, \quad \text{z lies in IV quadrant}$$

$$\tan\theta = -1 = \tan\left(2\pi - \frac{\pi}{4}\right) = \tan\frac{7\pi}{4}$$

$$\theta = \frac{7\pi}{4}$$

$\therefore$ Required form of Z is

$$Z = r(\cos\theta + i\sin\theta)$$

$$Z = \sqrt{2}[\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}]$$

Conversion from Polar form to Cartesian Form

Let $Z = r(\cos\theta + i\sin\theta)$ be the polar form and x + iy be its rectangular form.

Put $x = r\cos\theta$, $y = r\sin\theta$ to get required form.

Example 12. Convert 4(cos 300° + i sin 300°) into Cartesian from.

Sol. $4(\cos 300° + i \sin 300°) \xrightarrow{\text{change}} x + iy$

Put $x = 4\cos 300^o = 4\cos(360^o - 60^o) = 4\cos 60^o = 4 \times \frac{1}{2} = 2$

$y = 4\sin 300° = 4\sin(360° - 60°) = -4\sin 60° = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$

$\therefore$ Required Cartesian form is

$$x + iy = 2 - 2\sqrt{3}i$$

Modulus and Amplitude of a Complex Number

If z = x + iy is a complex number, then

a) $r = \sqrt{x^2 + y^2} = \sqrt{(\text{real part})^2 + (\text{Imaginary})^2}$ is known as Modulus of z.

it is denotes by | z |.

Thus Modulus of z = $|z| = r = \sqrt{x^2 + y^2}$

(b) Amplitude of z

$$\tan\theta = \frac{y}{x} \text{ i.e. } \tan\theta = \frac{-Im(z)}{Re(z)}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \text{ is known as Amplitude or argument of z.}$$

Example 13. Find the Modulus and Amplitude of a + ib.

Sol. Here z = a + ib

(i) Modulus, $|z| = \sqrt{a^2 + b^2}$

(ii) Amplitude, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Example 14. Find Modulus and amplitude of the complex number 1 + i.

Sol. Here z = 1 + I i.e. x = 1, y = 1

(i) Modulus $|z| = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$

(ii) Amplitude : $\tan\theta = \frac{y}{x} = \frac{1}{1}$

$$\tan\theta = 1 = \tan\frac{\pi}{4}$$

$$\theta = \frac{\pi}{4}$$

Example 15. Find the Modulus and Amplitude of the complex number –1 + i.

Sol. Let z = –1 + i

Compare it with z = x + iy, we get

x = –1, y = 1

(1) Modulus of z = $|z| = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

(2) Amplitude of z $\Rightarrow \tan\theta = \frac{y}{x} = \frac{1}{-1} = -1$

Complex number –1 +i lie in II quadrant.

θ also lie in IInd quadrant.

$$\tan\theta = -1 = \tan(180° - 45°) = \tan 135°$$

$\therefore$ $\theta = 135°$ or $\theta = \frac{3\pi}{4}$.

Example 16. Find modulus of each of the complex numbers 6 + 7i and 1 + 10i.

Sol. Let $z_1 = 6 + 7i$, $z_2 = 1 + 10i$

Then Modulus of $z_1 = |z_1| = \sqrt{6^2+7^2} = \sqrt{36+49} = \sqrt{85}$

Modulus of $z_2 = |z_2| = \sqrt{(1)^2+(10)^2} = \sqrt{1+100} = \sqrt{101}$

Example 17. Find modulus and amplitude of the complex number $-1+\sqrt{3}i$.

Solution : Let z = $=1+\sqrt{3}\,i$

then $x = -1,\ y = \sqrt{3}$

(1) Modulus of $z = |z| = \sqrt{(-1)^2+(\sqrt{3})^2} = \sqrt{1+3} = 2$

(2) Amplitude of z $\Rightarrow$ $\tan\theta = \dfrac{y}{x} = \dfrac{\sqrt{3}}{-1} = -\sqrt{3}$

Complex number lie in IInd quadrant.

$\therefore$ $$\tan\theta = -\sqrt{3} = \tan\left(\pi - \frac{\pi}{6}\right) = \tan\frac{5\pi}{6}.$$

$$\theta = \frac{5\pi}{6}.$$

Addition, Subtraction, Multiplication and Division of Complex Numbers :

(i) Addition of Complex Numbers : Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers.

Then $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$

$= (x_1 + x_2) + i(y_1 + y_2)$

i.e add real parts and Im. parts separately.

Note : Addition of two complex numbers is also a complex number.

Example 18. Let $z_1 = 7 + 3i$, $z_2 = 9 - i$ then

$$z_1 + z_2 = (7 + 3i) + (9 - i)$$

$$= (7 + 9) + i(3 - 1) = 16 + 2i$$

Note $\qquad z_1 + z_2 = z_2 + z_1$

(ii) Subtraction of Complex Numbers :

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex number.

then $\quad z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$

$$= (x_1 - x_2) + i(y_1 - y_2)$$

Note – Difference of two complex numbers is also complex number.

Example 19. Let $z_1 = 2 + 4i$, $z_2 = 7 + 5i$

then $\qquad z_1 - z_2 = (2 + 4i) - (7 + 5i)$

$$= (2 - 7) + i(4 - 5)$$

$$= -5 - i$$

Note $\qquad z_1 - z_2 \neq z_2 - z_1$

(iii) Multiplication of Complex Numbers

(a) In Cartesian form

Let $\quad z_1 = x + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then

$$z_1z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 \ (i^2 = -1)$$

$$z_1z_2 = (x_1 x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

(b) In polar form

Let $\qquad z_1 = r_1(\cos\theta_1 + i\sin\theta_2)$ and

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ are two complex number in polar form.

Then $\qquad z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

Note : Multiplication of two complex number is also a complex number.

(iv) Division of two complex numbers

(a) In Cartesian form

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}$$

$$= \frac{x_1x_3 - ix_1y_2 + ix_2y_1 - iy_1y_2}{x_2^2 - i^2y_2^2}$$

$$= \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \qquad (i^2 = -1)$$

$$\therefore \qquad \frac{z_1}{z_2} = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right)$$

Note : Division of two complex numbers is also a complex number.

(b) In Polar form

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + \sin\theta_1)}{r_2(\cos\theta_2 + \sin\theta_2)}$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Example 20. if $z_1 = -2 + 4i$ and $z_2 = 1 - 3i$, then find z_1z_2.

Sol. $z_1z_2 = (-2 + 4i)(1 - 3i)$

$= -2 + 6i + 4i - 12i^2 \qquad (i^2 = -1)$

$= -2 + 10i + 12$

$= 10 + 10i$

Example 21. if $z_1 = 5(\cos 30° + i\sin 30°)$ and $z_2 = 2(\cos 30° + i\sin 30°)$. Find z_1z_2.

Sol. $z_1z_2 = 5 \times 2[\cos(30 + 30) + i\sin(30 + 30)]$

$= 10[\cos 60° + i\sin 60°]$

$$= 10\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$$

$= 5 + 5\sqrt{3}\,i$

Example 22. If $z_1 = 2 - i$, $z_2 = 2 + i$, then

$$\frac{z_1}{z_2} = \frac{2-i}{2+i} = \frac{2-i}{2+i} \times \frac{2-i}{2-i}$$

$$= \frac{(2-i)^2}{(2)^2-(i)^2} = \frac{4+i^2-4i}{4+1}$$

$$= \frac{3-4i}{5} = \frac{3}{5} - \frac{4i}{5}$$

Example 23. If $z_1 = 5 + 7i$, $z_2 = 9 - 3i$, find $\frac{z_1}{z_2}$

Sol. $$\frac{z_1}{z_2} = \frac{5+7i}{9-3i} = \frac{5+7i}{9-3i} \times \frac{9+3i}{9+3i}$$

$$= \frac{45+15i+63i+21i^2}{(9)^2-(3i)^2}$$

$$= \frac{(45-21)+i(15+63)}{81+9} = \frac{24+78i}{90}$$

$$= \frac{24}{90} + \frac{78}{90}i$$

Example 24. If $z_1 = 50[\cos 50° + i \sin 50°]$ and $z_2 = 10[\cos(-10°) + i \sin(-10°)]$

then $$\frac{z_1}{z_2} = \frac{50}{10}[\cos(50-(-10)+i\sin[50-(-10)]$$

$$= 5[\cos 60° + i \sin 60°]$$

$$= 5\left[\frac{1}{2} + \frac{\sqrt{3}}{2}i\right]$$

$$= \frac{5}{2} + \frac{5\sqrt{3}}{2}i$$

Example 25. Find modulus of $z = 4i^3 + 3i^2 + 5i$

given $z = 4(-i) - 3 + 5i$

$$z = -4i - 3 + 5i$$

$$z = -3 + i$$

$$|z| = \sqrt{(-3)^2 + (1)^2} = \sqrt{9+1} = \sqrt{10}$$

(V) Multiplicative Inverse of a Complex Number

Let z = x + iy be a complex number then multiplicative inverse of z is $\frac{1}{z}$

i.e. $$\frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$$

$$= \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

Example 26. Find the multiplicative inverse (MI) of 1 – 2i.

Sol. MI of z is given by $$\frac{1}{z} = \frac{1}{1-2i}$$

$$= \frac{1}{1-2i} \times \frac{1+2i}{1+2i} = \frac{1+2i}{1+4}$$

$$= \frac{1}{5} + \frac{2}{5}i$$

Example 27. Find the value of x and y if 3x + (2x – y)i = 6 – 3i.

Sol. Equating real and imaginary part, we get

$$3x = 6 \quad \Rightarrow \quad x = 2$$

and $$2x - y = -3$$

$$2(2) - y = -3 \quad \Rightarrow \quad 4 - y = -3$$

$$-y = -7 \quad \Rightarrow \quad y = 7$$

Example 28. Express in complex form $\frac{2-i}{(1-2i)^2}$.

Sol. $$\frac{2-i}{(1-2i)^2} = \frac{2-i}{1+4i^2-4i} = \frac{2-i}{-3-4i}$$

$$= \frac{2-i}{-3-4i} \times \frac{-3+4i}{-3+4i} = \frac{-6+8i+3i-4i^2}{9+16}$$

$$= \frac{-2+11i}{25} = \frac{-2}{25} + \frac{11}{25}i \text{ which is required x + iy form}$$

Example 29. Simplify $\left(\frac{1}{2}+2i\right)^3$

Sol. $$\left(\frac{1}{2}+2i\right)^3 = \left(\frac{1}{2}\right)^3 + (2i)^3 + 3\left(\frac{1}{2}\right)(2i)\left[\frac{1}{2}+2i\right]$$

$$= \frac{1}{8} + 8i^3 + 3i\left(\frac{1}{2}+2i\right)$$

$$= \frac{1}{8} - 8i + \frac{3}{2}i + 6i^2$$

$$= \frac{1}{8} - 8i + \frac{3}{2}i - 6 = \frac{-47}{8} - \frac{13}{2}i$$

Example. 30. Find multiplicative inverse of $z = (6 + 5i)^2$.

Sol. $$z = (6+5i)^2 = 36 + 25i^2 + 60i$$

$$= 36 - 25 + 60i$$

$$= 11 + 60i$$

M.I. of $$z = \frac{1}{z} = \frac{1}{11+60i} = \frac{1}{11+60i} \times \frac{11-60i}{11-60i}$$

$$= \frac{11-60i}{121+3600} = \frac{11-60i}{3721}$$

$$= \frac{11}{3721} - \frac{60}{3721}i$$

Example 31. Express $\frac{1}{3+i} - \frac{1}{3-i}$ in x + iy form

Sol. $$\frac{1}{3+i} - \frac{1}{3-i} = \frac{3-i-3-i}{(3+i)(3-i)} = \frac{-2i}{9+1}$$

$$= \frac{0-2i}{10} \qquad = 0 - \frac{1}{5}i \text{ is the required form.}$$

Example 32. Express $\frac{(3+i)(4-i)}{5+i}$ in the form a + ib.

Sol.
$$\frac{(3+i)(4-i)}{5+i} = \frac{12-3i+4i-i^2}{5+i} = \frac{13+i}{5+i}$$

$$\frac{13+i}{5+i} \times \frac{5-i}{5-i} = \frac{65-13i+5i-i^2}{25+1} = \frac{66-8i}{26}$$

$$\frac{66}{26} - \frac{8}{26}i \qquad = \frac{33}{13} - \frac{4}{13}i$$ is required a + ib form.

Example 33. Find Modulus and amplitude of $\frac{1+i}{1-i}$.

Sol. Let
$$z = \frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i)^2}{1^2+i^2} = \frac{(1+i)^2}{1+1}$$

$$z = \frac{1+i^2+2i}{2} = \frac{1-1+2i}{2} = \frac{2i}{2} = i$$

z = i = 0 + i

(a) Modulus $|z| = \sqrt{(0)^2+(1)^2} = 1$

(b) Amplitude $= \tan\theta = \frac{y}{x} = \frac{1}{0} = \infty = \tan\frac{\pi}{2}$

$$\theta = \frac{\pi}{2}.$$

Example 34. Simplify $1+i^{100}+i^{10}+i^{50}$.

Sol.
$$1+i^{100}+i^{10}+i^{50} = 1+(i^4)^{25}+i^8.i^2+i^{48}.i^2$$

$$= 1+(1)^{25}+(i^4)^2 i^2+(i^4)^{12} i^2$$

$$= 1 + 1 + (1).i^2 + (1)\, i^2$$

$$= 1 + 1 - 1 - 1 = 0$$

EXERCISE –III

Questions on complex numbers:

1. If (x + iy) (2 – 3i) = 4 – I, find x and y.

2. If (a – 2bi) + (b – 3ai), find a and b.

3. Find real value of x and y if (1 – i)x + (1 + i)y = 1 – 3i.

4. Evaluate (i) i^{25} (ii) i^{19}

5. Find modulus and Amplitude of following

   (i) $1+\sqrt{3}i$

   (ii) 1 + i

   (iii) $4\sqrt{3}+4i$

   (iv) $z = \dfrac{3+2i}{4-5i}$

6. Add 2 + 3i and 5 – 6i

7. Subtract 7 – 5i from 2+4*i*

8. Simplify (5 + 5i) (4 – 3i)

9. If $z_1 = 1 + 3i$, $z_2 = 2 + i$, find z_1z_2

10. Write $\dfrac{3+4i}{2-3i}$ in x + iy form

11. Simplify $\dfrac{4-7i}{3-2i}$

12. Find multiplicative inverse of 3 + 4i

13. Write conjugate of –3 + 2i

14. Write conjugate of $\frac{3+2i}{4-5i}$

15. Express $\frac{(2+3i)^2}{1-i}$ is x + iy form

16. $z_1 = 2(\cos 60° + i \sin 60°)$, $z_2 = 4(\cos 15° + i \sin 15°)$, find $\frac{z_1}{z_2}$.

17. Write $\left(\frac{3}{2}+\frac{\sqrt{5}}{2}i\right)^2$ in x + iy form

18. Find modulus and conjugate of $\frac{2+7i}{3+2i}$

19. Express $\frac{3+2i}{2-5i}+\frac{3-2i}{2+5i}$ into x + iy form

20. Express $\frac{2+4i}{2-3i}$ in complex form x + *i*y

21. Write $\frac{(1-i)(2-i)(3-i)}{1+i}$ in x + iy form

22. Find the conjugate of $(3-7i)^2$

23. Find the amplitude of z if z = $\frac{-1-\sqrt{3}i}{2}$

24. The value of $\frac{i+i^2+i^3+i^4+i^5}{i+1}$ is

(a) $\frac{1-i}{2}$ (b) 1 (c) $\frac{1}{2}$ (d) $\frac{i+1}{2}$

25. If $(x+\sqrt{y})(p+\sqrt{q}) = x^2+y^2$ then

(a) p = x, q = y (b) $p = x^2$, $q = y^2$ (c) p = y, q = x (d) None of these

26. $(3a^{-2} + 2b^{-1})^2$ =

(a) $9a^{-4} + 4b^{-2} + 12a^{-2}b^{-1}$ (b) $9a^0 + 4b^0 + 12a^{-2}b^{-1}$

(c) $9a^{-2} + 4b^{-4} + 12a^{-2}b^{-1}$ (d) None of these

27. The value of $\sqrt{(-1)(-1)}$ is

(a) 1 (b) -1 (c) *i* (d) -*i*

28. The roots of the equation, $ax^2 + 2bx + c = 0$ are

(a) $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ (b) $\frac{-b\pm\sqrt{4b^2-4ac}}{2a}$

(c) $\frac{-2b \pm \sqrt{4b^2 - 4ac}}{2a}$ (d) None of these

29. The factorization of $x^2 - 5x + 6 = 0$ is
(a) (x+2) (x-3) (b) (x+2) (x+3)
(c) (x-2) (x-3) (d) (x-2) (x+3)

30. In a proper fraction;
(a) The degree of numeration is equal to the degree of denominator
(b) The degree of numeration is less than the degree of denominator
(c) The degree of numeration is more than the degree of denominator
(d) None of these

31. The partial fractions of $\frac{x+3}{x^2+x}$ are
(a) $\frac{2}{x} - \frac{3}{x+1}$ (b) $-\frac{2}{x} + \frac{3}{x+1}$
(c) $\frac{3}{x} - \frac{2}{x+1}$ (d) $\frac{3}{x} + \frac{2}{x+1}$

32. The multiplicative inverse of 1+i is
(a) $\frac{1}{2}(1 - i)$ (b) $\frac{1}{2}(1 + i)$
(c) (1 - i) (d) None of these

33. The real & imaginary parts of $\frac{(2+3i)^2}{1-i}$ are
(a) $\frac{22}{5}$ & $\frac{19}{5}$ (b) $\frac{-22}{5}$ & $\frac{-19}{5}$ 0028c) $\frac{-22}{5}$ & $\frac{19}{5}$ (d) None of these

34. Amplitude of $\frac{-1-\sqrt{3}\,i}{2}$ is
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{3}$

35. Conjugate of $(2 + i)^2$ is
(a) -3 + 4i (b) 3 - 4i (c) -3 - 4i (d) None of these

ANSWERS

1. $x = \frac{11}{3}$, $y = \frac{10}{3}$ 2. a = –12, b = 17 3. x = 0, y = –1 4. (i) i (ii) – i

5. (i) $r = 2, \theta = \frac{\pi}{3}$ (ii) $r = \sqrt{2}, \theta = \frac{\pi}{4}$ (iii) $r = 8, \theta = \frac{\pi}{3}$ (iv) $|z| = \frac{\sqrt{533}}{41}$, $\theta = \tan^{-1}\left(\frac{23}{2}\right)$

6. 7 – 3i 7. –5 + 9i 8. [35 + 5i] 9. –1 + 7i 10. $\frac{-6}{13} + \frac{17}{13}i$ 11. $10\left[\frac{1}{2} + \frac{\sqrt{3}}{2}i\right]$

12. $\left(\frac{3}{25}\right)+\left(\frac{-4}{25}i\right)$ 13. $-3-2i$ 14. $\frac{2}{41}-\frac{23}{41}i$ 15. $\frac{-22}{5}+\frac{19}{5}i$ 16. $-\frac{1}{2}\left[\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right]$

17. $1+\frac{3\sqrt{5}}{2}i$ 18. $|z|=1, \bar{z}=\frac{12}{13}-\frac{5}{13}i$ 19. $\frac{-8}{29}$ 20. $-\frac{6}{13}+\frac{17}{13}i$ 21. $5+5i$

22. $-40-42i$ 23. $\frac{\pi}{3}$ 24. (d) 25. (d) 26. (a) 27. (b)

28. (c) 29. (c) 30. (b) 31. (c) 32. (a) 33. (c)

34. (d) 35. (b)

1.4 LOGARITHM

Definition: If y and a are positive real numbers $(a \neq 1)$, then $x = log_a y$ if and only if $a^x = y$.

The notation $log_a y$ is read as "log to the base a of y". In the equation $x = log_a y$, x is known as the **logarithm**, a is the **base** and y is the **argument**.

<u>**Note**</u>: 1. The above definition indicates that a logarithm is an exponent.

Logarithm and Exponent

$x = log_a y$ if and only if $a^x = y$

Base

2. Logarithm of a number may be negative but the argument of logarithm must be positive. The base must also be positive and not equal to 1.

3.

Logarithmic Form	Exponential Form
$x = log_a y$	$a^x = y$

4. Logarithm of zero doesn't exist.
5. Logarithms of negative real numbers are not defined in the system of real numbers.
6. Log to the base "10" is called Common Logarithm and Log to the base "e" is called Natural Logarithm. $(e = 2.7182818284...)$
7. If base of logarithm is not given, it is considered to be Natural Logarithm.

Some Examples of logarithmic form and their corresponding exponential form:

S. No.	Logarithmic form	Exponential form
1	$5 = log_2 32$	$2^5 = 32$

2	$4 = log_3 81$	$3^4 = 81$
3	$3 = log_5 125$	$5^3 = 125$
4	$4 = log_{10} 10000$	$10^4 = 10000$
5	$-2 = log_7\left(\frac{1}{49}\right)$	$7^{-2} = \frac{1}{49}$
6	$0 = log_e 1$	$e^0 = 1$

Why do we study logarithms: Sometimes multiplication, subtraction and exponentiation become so lengthy and tedious to solve. Logarithms covert the problems of multiplication into addition, division into subtraction and exponentiation into multiplication, which are easy to solve.

Properties of Logarithms: If a, b and c are positive real numbers, $a \neq 1$ and n is any real number, then

1. **Product property:** $log_a(b.c) = log_a b + log_a c$

   For ex: $log_{10}(187) = log_{10}(11 \times 17) = log_{10}11 + log_{10}17$

2. **Quotient property:** $log_a\left(\frac{b}{c}\right) = log_a b - log_a c$

   For ex: $log_7\left(\frac{51}{7}\right) = log_7 51 - log_7 7$

3. **Power property:** $log_a b^n = n.log_a b$

   For ex: $log_{10}(10000) = log_{10}(10^4) = 4.log_{10}10$

4. **One to One property:** $log_a b = log_a c$ **if and only if** $b = c$**.**

   For ex: If $log_{10}(a) = log_{10}(15)$ then $a = 15$.

5. $log_a 1 = 0$

   For ex: $log_{10}(1) = 0$, $log_2(1) = 0$, $log_e(1) = 0$ etc.

6. $log_a a = 1$

   For ex: $log_{10}(10) = 1$, $log_e(e) = 1$ etc.

7. $log_a a^n = n$

   For ex: $log_{10}10^4 = 4$

8. $a^{log_a(n)} = n$**, where** $n > 0$

   For ex: $2^{log_2(8)} = 8$

9. **Change of base property:** $log_a b = \frac{log(b)}{log(a)} = \frac{log_c(b)}{log_c(a)}$ **provided that** $c \neq 1$**.**

For ex: $log_2(3) = \frac{log_{10}(3)}{log_{10}(2)}$ $(Here\ we\ changed\ the\ base\ to\ 10)$

Some solved examples:

Example 35. Convert the following exponential forms into logarithmic forms:

(i) $9^3 = 729$ **(ii)** $7^5 = 16807$ **(iii)** $2^{10} = 1024$

(iv) $10^{-3} = 0.001$ **(v)** $4^{-2} = 0.0625$ **(vi)** $5^{-4} = 0.0016$

(vii) $10^0 = 1$ **(viii)** $8^0 = 1$

Sol. (i) Given that $9^3 = 729$

$\Rightarrow$ $log_9 729 = 3$ $(by\ definition)$

which is required logarithmic form.

OR

Given that $9^3 = 729$

Taking logarithm on both sides, we get

$log 9^3 = log 729$

$\Rightarrow$ $3\ log 9 = log 729$ $(used\ power\ property)$

$\Rightarrow$ $3 = \frac{log 729}{log 9}$

$\Rightarrow$ $3 = log_9 729$ $\left(used\ log_a b = \frac{log(b)}{log(a)}\right)$

which is required logarithmic form.

(ii) Given that $7^5 = 16807$

$\Rightarrow$ $log_7 16807 = 5$ $(by\ definition)$

which is required logarithmic form.

OR

Given that $7^5 = 16807$

Taking logarithm on both sides, we get

$log 7^5 = log 16807$

$\Rightarrow$ $5\ log 7 = log 16807$ $(used\ power\ property)$

$\Rightarrow$ $5 = \frac{log 16807}{log 7}$

$\Rightarrow$ $5 = log_7 16807$ $\left(used\ log_a b = \frac{log(b)}{log(a)}\right)$

which is required logarithmic form.

(iii) Given that $2^{10} = 1024$

$\Rightarrow$ $log_2 1024 = 10$

which is required logarithmic form.

(iv) Given that $10^{-3} = 0.001$

$\Rightarrow$ $log_{10} 0.001 = -3$

which is required logarithmic form.

(v) Given that $4^{-2} = 0.0625$

$\Rightarrow$ $log_4 0.0625 = -2$

which is required logarithmic form.

(vi) Given that $5^{-4} = 0.0016$

$\Rightarrow$ $log_5 0.0016 = -4$

which is required logarithmic form.

(vii) Given that $10^0 = 1$

$\Rightarrow \qquad log_{10}1 = 0$

which is required logarithmic form.

(viii) Given that $8^0 = 1$

$\Rightarrow \qquad log_8 1 = 0$

which is required logarithmic form.

Example 36. Convert the following logarithmic forms into exponential forms:

(i) $log_\pi 1 = 0$ **(ii)** $log_2 2048 = 11$ **(iii)** $log_4\left(\frac{1}{64}\right) = -3$

(iv) $log_3 243 = 5$ **(v)** $log_{10}0.01 = -2$

Sol. (i) Given that $log_\pi 1 = 0$

$\Rightarrow \qquad \pi^0 = 1$

which is required exponential form.

(ii) Given that $log_2 2048 = 11$

$\Rightarrow \qquad 2^{11} = 2048$

which is required exponential form.

(iii) Given that $log_4\left(\frac{1}{64}\right) = -3$

$\Rightarrow \qquad 4^{-3} = \frac{1}{64}$

which is required exponential form.

(iv) Given that $log_3 243 = 5$

$\Rightarrow \qquad 3^5 = 243$

which is required exponential form.

(v) Given that $log_{10}0.01 = -2$

$\Rightarrow \qquad 10^{-2} = 0.01$

which is required exponential form.

Example 37. Evaluate the following:

(i) $log_2(8 \times 16)$ **(ii)** $log_4\left(\frac{16}{256}\right)$ **(iii)** $log_3 9^3$

(iv) $log_{10}\left(\frac{1}{10}\right)^8$ **(v)** $log_e\left(\frac{1}{e^{-7}}\right)$

Sol. (i) Given expression is

$$log_2(8 \times 16) = log_2(8) + log_2(16)$$

$(used\ product\ property)$

$$= log_2 2^3 + log_2 2^4$$

$$= 3.log_2 2 + 4.log_2 2 \qquad (used\ power\ property)$$

$$= 3 + 4 = 7 \qquad (used\ log_a a = 1)$$

which is required solution.

OR

Given expression is

$$log_2(8 \times 16) = log_2(128)$$

$$= log_2 2^7 = 7 \qquad (used\ log_a a^n = n)$$

which is required solution.

(ii) Given expression is

$$log_4\left(\frac{16}{256}\right) = log_4(16) - log_4(256) \qquad (used\ quotient\ property)$$

$$= log_4 4^2 - log_4 4^4$$

$$= 2.log_4 4 - 4.log_4 4 \qquad (used\ power\ property)$$
$$= 2 - 4 = -2 \qquad (used\ log_a a = 1)$$

which is required solution.

OR

Given expression is

$$log_4\left(\frac{16}{256}\right) = log_4\left(\frac{4^2}{4^4}\right)$$
$$= log_4 4^{-2} = -2 \qquad (used\ log_a a^n = n)$$

which is required solution.

(iii) Given expression is

$$log_3 9^3 = log_3 (3^2)^3$$
$$= log_3 3^6$$
$$= 6.log_3 3 \qquad (used\ power\ property)$$
$$= 6 \qquad (used\ log_a a = 1)$$

which is required solution.

(iv) Given expression is

$$log_{10}\left(\frac{1}{10}\right)^8 = log_{10}\frac{1}{10^8}$$
$$= log_{10} 1 - log_{10} 10^8 \qquad (used\ quotient\ property)$$
$$= 0 - 8.log_{10} 10 \qquad (used\ log_a 1 = 0)$$
$$= -8 \qquad (used\ log_a a = 1)$$

which is required solution.

(v) Given expression is

$$log_e\left(\frac{1}{e^{-7}}\right) = log_e e^7$$
$$= 7.log_e e$$
$$= 7 \qquad (used\ log_a a = 1)$$

which is required solution.

Example 38. Change the base of $log_2 3$ to 10 i.e. common logarithm**.**

Sol. Given expression is $log_2 3$

$$= \frac{log_{10}(3)}{log_{10}(2)}$$

Example 39. Change the base of $log_7 5$ to 5**.**

Sol. Given expression is $log_7 5$

$$= \frac{log_5(5)}{log_5(7)} = \frac{1}{log_5(7)}$$

Example 40. Solve the equation $log_2(x+1) = log_2(x) + log_2(2x+1)$ for x.

Sol. Given equation is

$$log_2(x+1) = log_2(x) + log_2(2x+1)$$
$$\Rightarrow \quad log_2(x+1) = log_2(x.(2x+1))$$
$$\Rightarrow \quad log_2(x+1) = log_2(2x^2 + x)$$
$$\Rightarrow \quad x + 1 = 2x^2 + x \qquad (used\ log_a b = log_a c\ iff\ b = c)$$
$$\Rightarrow \quad 2x^2 = 1$$
$$\Rightarrow \quad x^2 = \frac{1}{2}$$

$\Rightarrow \qquad x = \pm\sqrt{\frac{1}{2}}$

But x can't be negative as it is the argument of logarithm. Therefore, $= \sqrt{\frac{1}{2}}$.

Example 41. Solve the equation $log_5 a^2 = 1$ for a.

Sol. Given equation is

$$log_5 a^2 = 1$$

$$\Rightarrow \quad log_5 a^2 = log_5 5$$

$$\Rightarrow \quad a^2 = 5$$

$$\Rightarrow \quad a = \pm\sqrt{5}$$

Example 42. Solve the equation $log_{\frac{1}{2}}(y^2 - 1) = -1$ for y.

Sol. Given equation is

$$log_{\frac{1}{2}}(y^2 - 1) = -1$$

$$\Rightarrow \quad log_{\frac{1}{2}}(y^2 - 1) = -\,log_{\frac{1}{2}}\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad log_{\frac{1}{2}}(y^2 - 1) = log_{\frac{1}{2}}\left(\frac{1}{2}\right)^{-1}$$

$$\Rightarrow \quad log_{\frac{1}{2}}(y^2 - 1) = log_{\frac{1}{2}} 2$$

$$\Rightarrow \quad (y^2 - 1) = 2$$

$$\Rightarrow \quad y^2 = 3$$

$$\Rightarrow \quad y = \pm\sqrt{3}$$

Example 43. Prove that $log_b a \,.\, log_c b \,.\, log_a c = 1$ where a, b and c are positive and are not equal to 1.

Sol. $log_b a \,.\, log_c b \,.\, log_a c$

$$= \frac{loga}{logb} \cdot \frac{logb}{logc} \cdot \frac{logc}{loga} \quad (used\ change\ of\ base\ property)$$

$$= 1$$

Hence proved.

Example 44. Prove that $2log_2 4 + log_2 9 - log_2 6 = log_2 24$.

Sol. $2log_2 4 + log_2 9 - log_2 6$

$$= log_2 4^2 + log_2 9 - log_2 6$$

$$= log_2 16 + log_2 9 - log_2 6$$

$$= log_2(16 \times 9) - log_2 6$$

$$= log_2\left(\frac{16 \times 9}{6}\right)$$

$$= log_2 24$$

Hence proved.

Example 45. Prove that $log_{10} 12 - 2log_{10} 4 + 2log_{10} 6 = log_{10} 27$.

Sol. $log_{10} 12 - 2log_{10} 4 + 2log_{10} 6$

$$= log_{10} 12 - log_{10} 4^2 + log_{10} 6^2$$

$$= log_{10} 12 - log_{10} 16 + log_{10} 36$$

$$= log_{10}\left(\frac{12}{16}\right) + log_{10} 36$$

$$= log_{10}\left(\frac{12}{16} \times 36\right)$$
$$= log_{10}27$$

Hence proved.

EXERCISE -IV

1. Give the examples of following:
   (i) Product property (ii) Quotient property (iii) Power property.
2. Give the examples for following:
   (i) $log_a(b.c) \neq log_a b \times log_a c$
   (ii) $log_a\left(\frac{b}{c}\right) \neq \frac{log_a(b)}{log_a(c)}$
3. Convert the following exponential forms into logarithmic forms:
   (i) $5^5 = 3125$ (ii) $(0.2)^3 = 0.008$ (iii) $3^4 = 81$
   (iv) $2^{-3} = 0.125$ (v) $16^{-1} = 0.0625$ (vi) $e^0 = 1$
4. Convert the following logarithmic forms into exponential forms:
   (i) $log_{100}1 = 0$ (ii) $log_{10}0.0001 =- 4$ (iii) $log_e\left(\frac{1}{e}\right)^3 =- 3$
   (iv) $log_2 128 = 7$ (v) $log_{10}1000 = 3$
5. Evaluate the following:
   (i) $log_{10}\left(\frac{1}{10^{-12}}\right)$ (ii) $log_7\left(\frac{7^5}{49}\right)$ (iii) $log_3(3^5 \times 9^3)$
   (iv) $log_e\left(\frac{1}{e}\right)^{50}$ (v) $log_2\left(\frac{1}{1024}\right)$
6. Change the base of $log_5 9$ to 'e' i.e. natural logarithm.
7. Change the base of $log_7 19$ to '10' i.e. common logarithm.
8. Change the base of $log_3 11$ to 11.
9. Solve the following equation for x:
   (i) $log(2x^2 - 4) = log(2x) + log(x - 1)$
   (ii) $log(x^2) = log(40) - 3log2$
   (iii) $log(x) + log(6 + x) = log(16)$
   (iv) $log(x^2) = log(5x - 4)$
10. Prove that the following
    (i) $3log_2 4 + 4log_2 3 = log_2 5184$
    (ii) $3log4 - 2log2 + 3log5 = log80 + log25$
    (iii) $log_e e^6 + log_e\left(\frac{1}{e}\right)^3 = 3$
    (iii) $5log_2 2 - 2log_2 3 + log_2 18 = 6$

ANSWERS

3. (i) $log_5 3125 = 5$ (ii) $log_{0.2}0.008 = 3$ (iii) $log_3 81 = 4$
   (iv) $log_2 0.125 =- 3$ (v) $log_{16}0.0625 =- 1$ (vi) $log_e 1 = 0$
4. (i) $100^0 = 1$ (ii) $10^{-4} = 0.0001$ (iii) $e^{-3} = \left(\frac{1}{e}\right)^3$
   (iv) $2^7 = 128$ (v) $10^3 = 1000$

5. (i) 12 (ii) 3 (iii) 11 (iv) -50 (v) -10

6. $\frac{log_e 9}{log_e 5}$

7. $\frac{log_{10} 19}{log_{10} 7}$

8. $\frac{1}{log_3 11}$

9. (i) 2 (ii) $\pm\sqrt{5}$ (iii) 2 (iv) $1, 4$

1.5 DETERMINANTS AND MATRICES

Determinant : The arrangement of n^2 elements between two vertical lines in n rows and n–columns is called a determinant of order n and written as

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & \dots & a_{nn} \end{vmatrix}_{n\times n}$$

Here $a_{11}, a_{12}, a_{13}, \dots a_{nn}$ are called elements of determinant.

The horizontal lines are called rows and vertical lines are called columns.

Here a_{11} a_{12} $\dots$ a_{1n} $\rightarrow$ R_1 (First Row)

a_{21} a_{22} $\dots$ a_{2n} $\rightarrow$ R_2(IInd Row)

and

$$\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \dots & \dots \\ a_{n1} & a_{n2} \\ \downarrow & \downarrow \\ \text{Ist column} & \text{2nd column} \\ (C_1) & (C_2) \end{array}$$

Determinant of order 2: The arrangement of 4 elements in two rows and two columns between two vertical bar is called a determinant of order 2.

$$\text{i.e.} \quad \begin{array}{cc} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{array}{l} \rightarrow R_1 \text{ (first row)} \\ \rightarrow R_2 \text{ (2nd row)} \end{array} \\ \begin{array}{cc} C_1 & C_2 \\ \text{(1st)} & \text{(2nd)} \\ \text{Col.} & \text{Col.} \end{array} & \end{array}$$

Here a_{11}, a_{12}, a_{21}, a_{22} are called elements of determinant

Value of Determinant of order 2 :

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$ Multiplying diagonally with –sign in downward arrow

$$= a_{11}\, a_{22} - a_{21}\, a_{12}$$

Example 46. Solve $D = \begin{vmatrix} 2 & 5 \\ 3 & 9 \end{vmatrix}$.

Sol. $$D = \begin{vmatrix} 2 & 5 \\ 3 & 9 \end{vmatrix} = 2(9) - 3(5) = 18 - 15 = 3$$

Example 47. Find the value of, $D = \begin{vmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{vmatrix}$.

Sol. $$D = \begin{vmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{vmatrix} = \sin^2\theta - (-\cos^2\theta) = \sin^2\theta + \cos^2\theta = 1.$$

Determinant of 3rd Order : The arrangement of 3 × 3 = 9 elements between two vertical bars in 3–rows and 3 columns is called a determinant of order 3.

$$\text{i.e.} \quad \begin{array}{cc} D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{3} \end{vmatrix} & \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{array} \\ \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} & \end{array}$$

Here a_{11}, a_{12}, a_{13}, …, a_{33} are called elements of the determinant

Note : $a_{ij} \rightarrow$ element of i^{th} row and j^{th} column

$a_{mn} \rightarrow$ element of m^{th} row and n^{th} column

i.e. $a_{23} \rightarrow$ element of 2^{nd} row and 3^{rd} column.

Minor of an element

Definition : A minor of an element in a determinant in obtained by deleting row and column in which that element occurs.

Note: Minor of an element a_{ij} is denoted by M_{ij}

e.g.
$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Minor of an element $a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$$\begin{array}{cc} D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \text{eliminate} \\ \text{eliminate} & \end{array}$$

Minor of $a_{12} = M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Minor of $a_{13} = M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{32} & a_{33} \end{vmatrix}$

Minor of $a_{21} = M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

Similarly we can find minors of other elements.

Let Determinant of order 2 :

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\text{Minor of } a_{11} = M_{11} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{22}$$

Minor of $a_{12} = M_{12} = a_{21}$

Minor of $a_{21} = M_{21} = a_{12}$

Minor of $a_{22} = M_{22} = a_{11}$

Example 48. Find minor of each element in determinant $D = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$.

Minor of 1 = 4

Minor of 2 = 3

Minor of 3 = 2

Minor of 4 = 1

Example 49. Find minor of all the elements in the first row of the following determinant

$$D = \begin{vmatrix} 3 & 4 & -7 \\ -2 & 7 & 3 \\ 6 & -8 & 5 \end{vmatrix}$$

Sol. The elements in 1st row are 3, 4, –7.

(i) Minor of 3 = M_{11} (Deleting row and column in which element occurs)

i.e. $M_{11} = \begin{vmatrix} 7 & 3 \\ -8 & 5 \end{vmatrix} = 35 - (-24) = 59$

(ii) Minor of element 4 = $M_{12} = \begin{vmatrix} -2 & 3 \\ 6 & 5 \end{vmatrix} = -10 - 18 = -28$

(iii) Minor of 7 = $M_{13} = \begin{vmatrix} -2 & 7 \\ 6 & -8 \end{vmatrix}$ = 16 – 42 = –26.

Co–factor of an element

Co–factor of an element is a minor of that element with sign prefixed by the rule $(-1)^{i+j}$ where i and j are number of row and column in which that elements presents

e.g. $$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Co–factor of $a_{11} = (-1)^{1+1} M_{11} = a_{22}$

Co–factor of $a_{12} = (-1)^{1+2} M_{12} = -a_{21}$

Co–factor of $a_{21} = (-1)^{2+1} M_{21} = -a_{12}$

Co–factor of $a_{22} = (-1)^{2+2} M_{22} = a_{11}$

Example 50. Find the co–factor of element 5 and 2 from the Determinant $D = \begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix}$.

Sol. Co–factor of 5 = Co–factor of $a_{12} = (-1)^{1+2} M_{12} = (-1)(2) = -2$

Co–factor of 2 = co–factor of $a_{21} = (-1)^{2+1} M_{21} = (-1)5 = -5$.

Example 51. Find co–factor of a_{11} in $\begin{vmatrix} 2 & 7 \\ 5 & 3 \end{vmatrix}$.

Sol. Co–factor of $a_{11} = (-1)^{1+1} M_{11} = 3$.

Example 52. Find co–factor of all element a_{11}, a_{13}, a_{32}, a_{23} in the determinant $D = \begin{vmatrix} 3 & 4 & -7 \\ -2 & 7 & 3 \\ 6 & -8 & 5 \end{vmatrix}$.

Sol. Co–factor of $a_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} 7 & 3 \\ -8 & 5 \end{vmatrix} = 35 + 24 = 59$

Co–factor of $a_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} -2 & 7 \\ 6 & -8 \end{vmatrix} = 16 - 42 = -26$.

Co–factor of $a_{23} = (-1)^{2+3} M_{23} = -\begin{vmatrix} 3 & 4 \\ 6 & -8 \end{vmatrix} = -(-24-24) = 48$

Co–factor of $a_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} 3 & -7 \\ -2 & 3 \end{vmatrix} = -(9-14) = 5$

Similarly, we can find the cofactors of the remaining elements.

Evaluation of Determinant of 3 × 3 by Laplace expansion method :

A Determinant of order 3 × 3 is given by

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} \rightarrow R_1 \\ \\ \\ \end{matrix}$$

Expanding the determinant by First Row

$$= a_{11}[\text{Minor of } a_{11}] - a_{12}[\text{Minor of } a_{12}] + a_{13}[\text{Minor of } a_{13}]$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Example 53. Solve the following determinant by Laplace expansion method

$$D = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 5 & 2 \\ 1 & 8 & 2 \end{vmatrix} \begin{matrix} \rightarrow R_1 \\ \\ \\ \end{matrix}$$

Sol. Given determinant is

$$D = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 5 & 2 \\ 1 & 8 & 2 \end{vmatrix} \begin{matrix} \rightarrow R_1 \\ \\ \\ \end{matrix}$$

Expanding by R_1, we get

$= 3(\text{Minor of } 3) - 2(\text{Minor of } 2) + 1(\text{Minor of } 1)$

$$= 3\begin{vmatrix} 5 & 2 \\ 8 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 4 & 5 \\ 1 & 8 \end{vmatrix}$$

$= 3(10 - 16) - 2\,(8 - 2) + 1(8 - 5)$

$= 3(-6) - 2(6) + 1(3)$

$= -18 - 12 + 3 = -27$

Example 54. Find x if $\begin{vmatrix} 4 & x \\ x & 4 \end{vmatrix} = 0$.

Sol. Given that $\begin{vmatrix} 4 & x \\ x & 4 \end{vmatrix} = 0$

$\Rightarrow \quad 16 - x^2 = 0 \quad \Rightarrow \quad x^2 = 16, \; x = \pm 4$

Example 55. If $\begin{vmatrix} 3 & 2 \\ x & 6 \end{vmatrix} = 0$, find value of x.

Sol. Given that $\begin{vmatrix} 3 & 2 \\ x & 6 \end{vmatrix} = 0$

$\Rightarrow \quad 18 - 2x = 0 \quad \Rightarrow \quad -2x = -18, \quad \Rightarrow \quad x = 9$

Example 56. Evaluate by Laplace expansion Method

$$D = \begin{vmatrix} 5 & -1 & 2 \\ 1 & 3 & 1 \\ 7 & 1 & 0 \end{vmatrix} \begin{array}{l} \rightarrow R_1 \\ \\ \\ \end{array}$$

Sol. Expanding by R_1

$D = 5(\text{Minor of } 5) - (-1)\,[\text{Minor of } -1] + 2[\text{Minor of } 2]$

$$D = 5\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 7 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix}$$

$$D = 5(0 - 1) + 1(0 - 7) + 2(1 - 21)$$

$$D = -5 - 7 - 40 = -52$$

Solution of Equations by Cramer's Rule (having 2 unknown)

By Cramer's Rule, we can solve simultaneous equations with unknown using Determinants.

Solve the following equation by Cramer's rule

$$a_1x + b_1y = C_1 \qquad (1)$$

$$a_2x + b_2y = C_2 \qquad (2)$$

Solution of eqn. (1) and (2) by Cramer's Rule is

$$x = \frac{D_1}{D}, \quad y = \frac{D_2}{D}$$

where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

$$D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \text{ (Replacing first column by constants)}$$

$$D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \text{ (Replacing by 2}^{\text{nd}} \text{ column by constants)}$$

Note : The given equation have unique solution if $D \neq 0$.

Solution of Equation (in 3 unknown) by Cramer's Rule

Let three equations in three variable x, y, z be

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

The solution of given equation is

$$x = \frac{D_1}{D}, \ y = \frac{D_2}{D}, \ z = \frac{D_3}{D}$$

where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \text{(Determinant formed by coefficient of x, y, z)}$$

$$D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \quad \text{(obtained by replacing the 1st column by constant terms)}$$

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \quad \text{(obtained by replacing the 2nd column by constant terms)}$$

$$D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \quad \text{(obtained by replacing the 3rd column by constant terms)}$$

Now $\quad x = \frac{D_1}{D}, \ y = \frac{D_2}{D}, \ z = \frac{D_3}{D}$

Note : (1) The equations have unique solution, if $D \neq 0$.

(2) The equations have an infinite number of solutions if $D = D_1 = D_2 = D_3 = 0$.

(3) The equations have no solution if $D = 0$ and any one of D_1, D_2 and D_3 is not zero.

Consistant : When a system of equations have a solution, then equations are said to be consistant.

Inconsistent : If equations have no solution, then equations are said to be inconsistent.

Example 57. Solve the system of equations using Cramer's rules :

$$x + 2y = 1$$

$$3x + y = 4$$

Sol. $$D = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 - 6 = -5$$

$$D_1 = \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} = 1 - 8 = -7$$

$$D_2 = \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} = 4 - 3 = 1$$

$\therefore$ $$x = \frac{D_1}{D} = \frac{-7}{-5} = \frac{7}{5}$$

$$y = \frac{D_2}{D} = \frac{1}{-5} = -\frac{1}{5}$$

Example 58. Solve by Cramer's rule

$$10x + 10y - z = -2$$

$$x + 5y + 2z = 0$$

$$x - 5y - z = 4$$

Sol. The coefficient determinant is

$$D = \begin{vmatrix} 10 & 10 & -1 \\ 1 & 5 & 2 \\ 1 & -5 & -1 \end{vmatrix} \begin{matrix} \to R_1 \\ \\ \\ \end{matrix} \qquad \text{Expanding by } R_1$$

$$= 10(-5 + 10) - 10(-1 - 2) - 1(-5 - 5)$$

$$= 50 + 30 + 10 = 90$$

$$D_1 = \begin{vmatrix} -2 & 10 & -1 \\ 0 & 5 & 2 \\ 4 & -5 & -1 \end{vmatrix} \begin{matrix} \to R_1 \\ \\ \\ \end{matrix} \qquad \text{Expanding by } R_1$$

$$= -2(-5 + 10) - 10(0 - 8) - 1(0 - 20)$$

$$= -10 + 80 + 20 = 90$$

$$D_2 = \begin{vmatrix} 10 & -2 & -1 \\ 1 & 0 & 2 \\ 1 & 4 & -1 \end{vmatrix} \begin{matrix} \to R_1 \\ \\ \\ \end{matrix} \qquad \text{Expanding by } R_1$$

$$= 10(0 - 8) + 2(-1 - 2) - 1(4 - 0)$$

$$= -80 - 6 - 4 = -90$$

$$D_3 = \begin{vmatrix} 10 & 10 & -2 \\ 1 & 5 & 0 \\ 1 & -5 & 4 \end{vmatrix} \rightarrow R_1 \qquad \text{Expanding by } R_1$$

$$= 10(20 + 0) - 10(4 - 0) - 2(-5 + 5)$$

$$= 200 - 40 + 20 = 180$$

$$\therefore \qquad x = \frac{D_1}{D} = \frac{90}{90} = 1$$

$$y = \frac{D_2}{D} = \frac{-90}{+90} = -1$$

$$z = \frac{D_3}{D} = \frac{180}{90} = 2$$

Hence x = 1, y = –1, z = 2.

Example 59. Apply Cramer's Rule to solve the equations:

$$x - 2y + z = 1$$

$$2x + 3y + 2z = 2$$

$$-x + y + 3z = -1$$

Sol. The coefficient Determinant is

$$D = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & 2 \\ -1 & 1 & 3 \end{vmatrix}$$

$$= 1(9 - 2) - (-2)(6 + 2) + 1(2 + 3)$$

$$= 7 + 16 + 5 = 28$$

$$D_1 = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & 2 \\ -1 & 1 & 3 \end{vmatrix}$$

$$= 1(9 - 2) + 2(6 + 2) + 1(2 + 3)$$

$$= 7 + 16 + 5 = 28$$

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & 3 \end{vmatrix}$$

$$= 1(6 + 2) - 1(6 + 2) + 1(-2 + 2)$$

$$= 8 - 8 + 0 = 0$$

$$D_3 = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{vmatrix}$$

$$= 1(-3 -2) + 2(-2 + 2) + 1(2 + 3)$$

$$= -5 + 0 + 5 = 0$$

$\therefore$ $$x = \frac{D_1}{D} = \frac{28}{28} = 1$$

$$y = \frac{D_2}{D} = \frac{0}{28} = 0$$

$$z = \frac{D_3}{D} = \frac{0}{28} = 0$$

Hence solution is x = 1, y = 0, z = 0.

EXERCISE -V

1. Find minors of all elements in $\begin{vmatrix} 7 & -3 \\ 4 & 2 \end{vmatrix}$.

2. Find minors and co–factors of all elements of determinant $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$.

3. Evaluate $\begin{vmatrix} 2 & 4 \\ -5 & 1 \end{vmatrix}$.

4. Evaluate $\begin{vmatrix} 1 & \sin\theta \\ \sin\theta & 1 \end{vmatrix}$

5. Expand the determinant by Laplace Method $D = \begin{vmatrix} 5 & -1 & 2 \\ 0 & 1 & 3 \\ 2 & 3 & 4 \end{vmatrix}$

6. Evaluate $\begin{vmatrix} 5 & -1 & 2 \\ 1 & 3 & 1 \\ 7 & 1 & 0 \end{vmatrix}$.

7. Find x if $\begin{vmatrix} 4 & x \\ x & 4 \end{vmatrix} = 0$.

8. Find x if $\begin{vmatrix} 4 & 3 & 9 \\ 3 & 2 & 7 \\ 1 & 4 & x \end{vmatrix} = 0$

9. Solve by Cramer's rule $x + 3y = 4$

$4x - y = 3$

10. Solve by Determinant (Cramer's Rule) :

$x + y + 2z = 4$

$2x - y + 2z = 9$

$3x - y - z = 2$

11. Solve by Cramer's Rule :

$x + y - z = 0$

$2x + y + 3z = 9$

$x - y + z = 2$

12. Find Minors and Co–factors of 2nd row in elements of the Determinant.

$$\begin{vmatrix} 1 & 0 & 1 \\ -2 & 1 & 2 \\ 5 & 4 & -3 \end{vmatrix}$$

ANSWERS

1. $M_{11} = 2$, $M_{12} = 4$, $M_{21} = -3$, $M_{22} = 7$

2. $M_{11} = 3$, $M_{12} = 0$, $M_{21} = -4$, $M_{22} = 2$; $C_{11} = 3$, $C_{12} = 0$, $C_{21} = 4$, $C_{22} = 2$

3. 18 4. $\cos^2\boldsymbol{\theta}$ 5. –35 6. –52 7. $x = \pm 4$ 8. $x = -1$

9. $x = 1, y = 1$ 10. $x = 1, y = -1, z = 2$ 11. $x = 1, y = 1, z = 2$ 12. –4, –8, 4

MATRICES

Matrix: The arrangement of $m \times n$ elements in m–row and n–columns enclosed by a pair of brackets [ ] is called a matrix of order $m \times n$. The matrix is denoted by capital letter A, B, C etc.

A matrix of order $m \times n$ is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}$$

i.e. Matrix have m rows and n columns. In short we can write it as

$$A = [a_{ij}]$$

where $i = 1, 2, 3, \dots, m$ ← row

$j = 1, 2, 3, \dots, n$ ← columns

Note – Difference between a determinant & a matrix is that a determinant is always in square form [i.e. m = n], but matrix may be in square or in rectangular form. Determinant has a definite value, but matrix is only arrangement of elements with no value.

Order of Matrix : Number of rows × number of column

e.g. $A = \begin{bmatrix} 2 & 1 \\ 2 & 5 \end{bmatrix}_{2\times 2}$ is a matrix of order 2×2

$$B = \begin{bmatrix} 2 & 5 \\ 1 & 6 \\ -2 & 0 \end{bmatrix}_{3\times 2} \quad \text{is a matrix of order } 3 \times 2$$

Types of Matrices :

(1) Square Matrix : A matrix is said to be a square matrix if number of rows of matrix is equal to number of columns of Matrix i.e. $m = n$.

For ex (i) $\begin{bmatrix} 2 & 0 \\ 5 & 7 \end{bmatrix}_{2\times 2}$ is a square matrix of order 2×2.

(ii) $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 0 & 1 \\ 0 & 5 & 6 \end{bmatrix}_{3\times 3}$ is a square matrix of order 3×3

(2) Rectangular Matrix : A matrix where, number of rows is not equal to number of columns i.e. $m \neq n$

e.g. $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 0 & 6 \end{bmatrix}_{2\times 3}$ is a rectangular matrix of order 2×3

(3) Row Matrix : A matrix having one row and any number of columns is called row matrix

e.g. (i) $A = [-3 \quad 2]_{1 \times 2}$ isarow matrix of order 1×2

(ii) $B = [5 \quad 7 \quad -2]_{1 \times 3}$ is a row matrix of order 1×3

(4) Column Matrix : A matrix having only one column and any number of rows is called column matrix.

e.g. $A = \begin{bmatrix} 2 \\ 5 \end{bmatrix}_{2\times 1}$ order of matrix is 2×1

$B = \begin{bmatrix} 3 \\ 1 \\ 8 \end{bmatrix}_{3\times 1}$ order of matrix is 3×1

(5) Diagonal Matrix : A matrix is to be a diagonal matrix if all non−diagonal elements are zero.

e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times3}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times3}$

(6) Null Matrix : A matrix whose all elements are zero is called a null matrix. It is denoted by 0.

e.g. $0_{2x2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $0_{3x3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(7) Unit matrix : A diagonal matrix each of whose diagonal element is equal to unity is called unit matrix

For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are unit matrices of order 2 and 3 respectively.

(8) Scalar Matrix : A diagonal matrix is said to be scalar matrix of all diagonal elements are equal.

e.g. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(9) Upper Triangular Matrix : A square matrix is which all the elements below the principle diagonal are zero is called an upper triangular matrix.

e.g. $\begin{bmatrix} 2 & 3 & 7 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

(10) Lower Triangular Matrix : A square matrix in which all the elements above the principal diagonal are zero is called lower triangular matrix.

e.g. $$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 5 & 0 \\ 1 & 3 & 2 \end{bmatrix}$$

(11) Equal Matrix : Two matrices are said to equal if they have same order and their corresponding elements are identical

For example $$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

If $x_1 = 2$, $x_2 = 4$, $x_3 = 6$, $x_4 = 8$

(12) Transpose of Matrix : A matrix obtained by interchanging its rows and columns is called Transpose of the given matrix. Transose of A is denoted by A^T or A'.

e.g. If $$A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 5 & 6 \end{bmatrix}$$

then $$A^T = \begin{bmatrix} 2 & 2 \\ 3 & 5 \\ 1 & 6 \end{bmatrix}$$

Note $(A^T)^T = A$.

(13) Symmetric Matrix : A matrix is said to be symmetric if it is equal to its transpose i.e. $A^T = A$.

For ex $$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$ is a symmetric matrix

(14) Skew–symmetric matrix : A square matrix is said to be skew–symmetric is

$$A^T = -A$$

Note : The diagonal elements of skew–symmetric matrix are always be zero.

For ex $$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}$$ is skew symmetric matrix

(15) Singular Matrix: A square matrix is said to be singular if $|A| = 0$. *i.e.* determinant, where $|A|$ is the determinant of matrix A.

(16) Non–singular matrix : A matrix is said to be non–singular if $|A| \neq 0$.

Formation of a Matrix/Construction of Matrix :

Example 60. Construct a 2×2 matrix whose elements are $a_{ij} = i + j$.

Sol. We have

$$a_{ij} = i + j$$

Required matrix of 2×2 is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

Example 61. Construct a matrix of 2×2 whose element is given by. $a_{ij} = \frac{(i+2j)^2}{2}$

Sol. We have

$\therefore$ $$a_{ij} = \frac{(i+2j)^2}{2}$$

$$a_{12} = \frac{(1+4)^2}{2} = \frac{25}{2}$$

$$a_{21} = \frac{(2+2)^2}{2} = \frac{16}{4} = 8$$

$$a_{22} = \frac{(2+2(2))^2}{2} = \frac{36}{2} = 18$$

$\therefore$ $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{9}{2} & \frac{25}{2} \\ 8 & 18 \end{bmatrix}$$

Example 62. If $\begin{bmatrix} a+b & 2 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$ find the values of a and b.

Sol. Given matrices are equal, therefore their corresponding elements are identical

$\therefore$ $a + b = 6 \Rightarrow a = 6 - b$

and $ab = 8$

$\Rightarrow (6 - b)b = 8$

$6b - b^2 = 8 \Rightarrow -b^2 + 6b - 8 = 0$

$\Rightarrow b^2 - 6b + 8 = 0 \Rightarrow (b - 2)(b - 4) = 0$

$\Rightarrow b = 2$ and $b = 4$

If $b = 2 \Rightarrow a = 6 - b = 6 - 2 = 4$

If $b = 4 \Rightarrow a = 6 - b = 6 - 4 = 2$

$\therefore$ $a = 2,\ b = 4$ & $a = 4, b = 2$

Example. 63. Find the value of x, y, z and a if $\begin{bmatrix} 2x+3 & x+2y \\ z+3 & 2a-4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$.

Sol. given matrices are equal

$\therefore$ $2x + 3 = 1$ (i)

$x + 2y = 2$ (ii)

$z + 3 = -1$ (iii)

$2a - 4 = 3$ (iv)

From eqn. (i) $2x = 1 - 3 = -2$

$2x = -2 \Rightarrow x = -1$

Substitute $x = -1$ in (ii)

$-1 + 2y = 2$

$2y = 3 \Rightarrow y = \frac{3}{2}$

From eqn. (iii)

$$z + 3 = -1 \quad \Rightarrow \quad z = -4$$

From eqn. (iv)

$$2a - 4 = 3$$

$$2a = 7 \quad \Rightarrow \quad a = \frac{7}{2}$$

$$\therefore \qquad x = -1,\ y = \frac{3}{2},\ z = -4,\ a = \frac{7}{2}$$

Algebra of Matrices : (Addition, Subtraction, and Multiplication of Matrices)

Addition of Matrices : If A and B are two matrices having same order, then their addition A + B is obtained by adding there corresponding elements

For example, If

$$A = \begin{bmatrix} 2 & 5 \\ 6 & 8 \\ 7 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & 7 \\ 0 & 6 \\ 2 & 3 \end{bmatrix}$$

Then
$$A + B = \begin{bmatrix} 2+5 & 5+7 \\ 6+0 & 8+6 \\ 7+2 & 0+3 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 6 & 14 \\ 9 & 3 \end{bmatrix}$$

Properties of Matrix addition :

If A, B and C are three matrix of same order then

(i) A + B = B + A (commutative law)

(ii) A + (B + C) = (A + B) + C [Associative law)

(iii) A + 0 = 0 + A, where **0** is null matrix

(iv) A + (–A) = (–A) + A = **0**

Here (–A) is called additive inverse of matrix A.

Subtraction of Matrices : If A and B are two matrices of same order, then A − B is obtained by subtracting element of B from the corresponding elements of A.

For example, Let $A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \\ 2 & 0 \end{bmatrix}$

then $A - B = \begin{bmatrix} 5-3 & 2-5 \\ 3-2 & 2-0 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$

Note : $A - B \neq B - A$.

Scalar Multiplication : The matrix obtained by multiplying each element of a given matrix by a scalar K.

If $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$, then scalar multiplication of A by the scalar 2 is given by

$$2A = 2\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 8 \end{bmatrix}$$

Multiplication of Two Matrices :

If A and B are two matrices, then their product AB is possible only **if number of columns in A is equal to number of rows in B.**

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$

Then $C = [c_{ij}]_{m \times p}$ is called the product of A and B.

Example 64. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}_{2\times 2}$, $B = \begin{bmatrix} 6 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}_{2\times 3}$ then find the product AB.

Sol. Here no. of column in A = No. of row in B.

Hence AB exist

$$AB = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 6 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \end{bmatrix}$$

R_1C_1, R_1C_2, R_1C_3, similarly R_2C_1, R_2C_2 & R_3C_3

$$AB = \begin{bmatrix} 2(6)+3(3) & 2(0)+3(1) & 2(1)+3(2) \\ 4(6)+5(3) & 4(0)+5(1) & 4(1)+5(2) \end{bmatrix}$$

$$= \begin{bmatrix} 12+9 & 0+3 & 2+6 \\ 24+15 & 0+5 & 4+8 \end{bmatrix} = \begin{bmatrix} 21 & 3 & 8 \\ 39 & 5 & 12 \end{bmatrix}$$

Note : $AB \neq BA$ (In general)

Properties of Multiplication of Matrices

(i) $AB \neq BA$ (in general)

(ii) $A(B + C) = AB + AC$

(iii) $AI = A$, where I is the unit matrix.

Power of a Matrix : If A is a square matrix *i.e.* number of rows = number of its columns then,

$A^2 = A.A$

$A^3 = A^2.A = A.A^2$

Similarly we can find other power of square matrix.

e.g. if $A = \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}$, then find A^2

$$A^2 = A.A = \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-4 & 2+0 \\ -2+0 & -4+0 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -2 & -4 \end{bmatrix}$$

Example 65. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$, then find 5A and 3B.

Sol. We have $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$, then

$$5A = 5\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ 20 & 35 \end{bmatrix}$$

$$3B = 3\begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 12 & 18 \end{bmatrix}$$

Example 66. If $A = \begin{bmatrix} 7 & 3 \\ -5 & 7 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 4 \\ 5 & 8 \end{bmatrix}$, then find A – B.

Sol. $$A - B = \begin{bmatrix} 7 & 3 \\ -5 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 9 & -1 \\ -10 & -1 \end{bmatrix}$$

Example 67. If $A = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, then find 2A – 3B.

Sol. We have

$$A = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix},\ 2A = 2\begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ -2 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix},\ 3B = 3\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 9 & 6 \end{bmatrix}$$

$$2A - 2B = \begin{bmatrix} 10-6 & 6-(-3) \\ -2-9 & 2-6 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ -11 & -4 \end{bmatrix}$$

Example 68. If $X = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix}$. Find 3X + Y.

Sol. $$3X = 3\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -9 & 12 \end{bmatrix}$$

$$3X + Y = \begin{bmatrix} 3 & 6 \\ -9 & 12 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3+4 & 6+5 \\ -9+1 & 12-3 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ -8 & 9 \end{bmatrix}$$

Example 69. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$, then find 2A + 3B + 5I, where I is a unit matrix of order 2.

Sol. $$2A = 2\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 8 & 14 \end{bmatrix}$$

$$3B = 3\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ -6 & 15 \end{bmatrix}$$

$$5I = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Now $$2A + 3B + 5I = \begin{bmatrix} 4 & 6 \\ 8 & 14 \end{bmatrix} + \begin{bmatrix} 3 & 9 \\ -6 & 15 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4+3+5 & 6+9+0 \\ 8-6+0 & 14+15+5 \end{bmatrix} = \begin{bmatrix} 12 & 15 \\ 2 & 34 \end{bmatrix}$$

Example 70. If $A = \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 6 \\ -4 & -12 \end{bmatrix}$, find AB.

Sol. $$AB = \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ -4 & -12 \end{bmatrix} = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{bmatrix}$$

$$= \begin{bmatrix} 8-8 & 24-24 \\ 16-16 & 48-48 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example 71. if $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$, show that AB = BA.

Sol. $$AB = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2(3)+5(-1) & 2(-5)+5(2) \\ 1(3)+3(-1) & 1(-5)+3(2) \end{bmatrix}$$

$$= \begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6-5 & 15-15 \\ -2+2 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ AB = BA

Example 72. If $A=\begin{bmatrix}1 & 2\\ -2 & 0\end{bmatrix}$, $B=\begin{bmatrix}-3 & 2\\ -2 & 4\end{bmatrix}$, show that $(A + B)^2= A^2 + 2AB + B^2$.

Sol. $A+B=\begin{bmatrix}1 & 2\\ -2 & 0\end{bmatrix}+\begin{bmatrix}-3 & 2\\ -2 & 4\end{bmatrix}=\begin{bmatrix}-2 & 4\\ -4 & 4\end{bmatrix}$

LHS $(A + B)^2 = (A + B)(A + B) = \begin{bmatrix}-2 & 4\\ -4 & 4\end{bmatrix}\begin{bmatrix}-2 & 4\\ -4 & 4\end{bmatrix}$

$$= \begin{bmatrix}-12 & 8\\ -8 & 0\end{bmatrix} \qquad (1)$$

$$A^2 = A.A. = \begin{bmatrix}1 & 2\\ -2 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ -2 & 0\end{bmatrix}=\begin{bmatrix}1-4 & 2+0\\ -2+0 & -4+0\end{bmatrix}$$

$$= \begin{bmatrix}-3 & 2\\ -2 & -4\end{bmatrix}$$

$$A.B = \begin{bmatrix}1 & 2\\ -2 & 0\end{bmatrix}\begin{bmatrix}-3 & 2\\ -2 & 4\end{bmatrix}=\begin{bmatrix}-3-4 & 2+8\\ 6+0 & -4+0\end{bmatrix}=\begin{bmatrix}-7 & 10\\ 6 & -4\end{bmatrix}$$

$$2AB = 2\begin{bmatrix}-7 & 10\\ 6 & -4\end{bmatrix}=\begin{bmatrix}-14 & 20\\ 12 & -8\end{bmatrix}$$

$$B^2 = B.B. = \begin{bmatrix}-3 & 2\\ -2 & 4\end{bmatrix}\begin{bmatrix}-3 & 2\\ -2 & 4\end{bmatrix}=\begin{bmatrix}+9-4 & -6+8\\ 6-8 & -4+16\end{bmatrix}$$

$$= \begin{bmatrix}5 & 2\\ -2 & 12\end{bmatrix}$$

$\therefore$ RHS $A^2 + 2AB + B^2$

$$= \begin{bmatrix}-3 & 2\\ -2 & -4\end{bmatrix}+\begin{bmatrix}-14 & 20\\ 12 & -8\end{bmatrix}+\begin{bmatrix}5 & 2\\ -2 & 12\end{bmatrix}$$

$$= \begin{bmatrix}-3-14+5 & 2+20+2\\ -2+12-2 & -4-8+12\end{bmatrix}$$

$$= \begin{bmatrix}-12 & 24\\ 8 & 0\end{bmatrix}$$

$\therefore$ LHS $\neq$ RHS

Example 73. If $A = \begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 5 \\ 4 & 5 \end{bmatrix}$. Verify $(AB)^T = B^T A^T$.

Sol. Given matrix $A = \begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 5 \\ 4 & 5 \end{bmatrix}$, then $A^T = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix}$, $B^T = \begin{bmatrix} -3 & 4 \\ 5 & 5 \end{bmatrix}$.

$$AB = \begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} -3 & 5 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -9+16 & 15+20 \\ -3+20 & 5+25 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 35 \\ 17 & 30 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 7 & 17 \\ 35 & 30 \end{bmatrix} \qquad (1)$$

$$B^T A^T = \begin{bmatrix} -3 & 4 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -9+16 & -3+20 \\ 15+20 & 5+25 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 17 \\ 35 & 30 \end{bmatrix} \qquad (2)$$

From eqn.. (1) and (2)

$$(AB)^T = B^T A^T$$

Example 74. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, then find A^2.

Sol. $$A^2 = A.A. = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4 & 2+6 \\ 2+6 & 4+9 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}.$$

Example 75. If $A = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}$. Show that $A^2 - 5A + 5I = 0$, where I is unit matrix of order 2.

Sol. We have $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 9+1 & 3+2 \\ 3+2 & 1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 5 \\ 5 & 5 \end{bmatrix}$$

$$5A = 5\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ 5 & 10 \end{bmatrix}$$

$$5I = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

LHS $\quad A^2 - 5A + 5I$

$$\begin{bmatrix} 10 & 5 \\ 5 & 5 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ 5 & 10 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 10-15+5 & 5-5+0 \\ 5-5+0 & 5-10+5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 = \text{RHS}$$

EXERCISE-VI

1. Find the order of following matrices also find their type

(a) $[1 \quad 5 \quad 7]$ (b) $\begin{bmatrix} 6 \\ 9 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 0 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2. If $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$, find the value of x, y, z and w.

3. If $\begin{bmatrix} x+y & y-z \\ z-2x & y-x \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix}$, find x, y, z.

4. Construct a 2×2 matrix whose element $a_{ij} = \dfrac{(1+j)^2}{2}$.

5. If $A = \begin{bmatrix} 3 & 1 \\ 5 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$, find A + 2B.

6. Find the value of x, y, z and a if $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$.

7. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ -1 & 3 & 4 \end{bmatrix}$. Find $A^2 - 4A + 8I$, where I is unit matrix of order 3×3.

8. If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$. Show that AB = BA = I.

9. For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$. Find the number a and b such that $A^2 + aA + bI = 0$, where I is unit matrix of 2×2.

10. If $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$. Show that AB = BA.

11. If $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 \\ 4 & 5 \end{bmatrix}$, then evaluate AB + 2I.

12. If $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, then verify that

   (i) A(B + C) = AB + AC

   (ii) (B + C)A = BA + CA

13. If $A = \begin{bmatrix} 1 & 3 \\ x & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 4 & -3 \\ -2 & 3 \end{bmatrix}$. Verify that (A + B)C = AC + BC.

14. If $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, then find AB.

15. If $X = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix}$. Find 3X + Y.

16. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$, find :

(i) 2A + 3B – 4I, when I is a unit matrix

(ii) 3A – 2B

17. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$, find 2A + 3B – 5I, where I is unit matrix.

18. - The value of $\begin{vmatrix} 1 & 2 & 3 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix}$ is

(a) x^2 (b) 2x (c) 6x (d) x

19. - The value of $\begin{vmatrix} 0 & -c & -b \\ c & 0 & -a \\ b & a & 0 \end{vmatrix}$ is

(a) 1 (b) a + b + c (c) 0 (d) abc

20 - Cofactors of the first row elements is the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ are

(a) 3, -6, 3 (b) -3, 6, -3 (c) 3, 6, 3 (d) -3, -6, -3

21 - Two matrices A_{mxn} & B_{pxq} can be multiplied only when

(a) m = p (b) n = p (c) m = q (d) n = q

22 - If $A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$ then A^2 is

(a) $\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ (d) None of these

ANSWERS

1. (a) order 1 × 3, Row matrix

(b) order 3 × 1, Column matrix

(c) order 2 × 2, Scalar matrix

(d) order 2 × 2, Square matrix

(e) order 2 × 3, Rectangular matrix

(f) order 2×2, Null matrix

2. $x = 2, y = 4, z = 1$, and $w = 3$ 3. $x = 2, y = 1, z = 2$

4. $\begin{bmatrix} 2 & \frac{9}{2} \\ \frac{9}{2} & 8 \end{bmatrix}$ 5. $\begin{bmatrix} -3 & 5 \\ 1 & 3 \end{bmatrix}$ 6. $x = -3, y = -2, z = 4, a = 3$

7. $\begin{bmatrix} 12 & 15 & 17 \\ 8 & 61 & 57 \\ 14 & 16 & 26 \end{bmatrix}$ 11. $\begin{bmatrix} 4 & 1 \\ 13 & 16 \end{bmatrix}$ 14. $\begin{bmatrix} a^2+b^2 & 0 \\ 0 & b^2+a^2 \end{bmatrix}$

15. $\begin{bmatrix} 7 & 11 \\ -8 & 9 \end{bmatrix}$ 16.(i) $\begin{bmatrix} 3 & 15 \\ 20 & 28 \end{bmatrix}$, (ii) $\begin{bmatrix} 4 & 3 \\ 4 & 9 \end{bmatrix}$ 17. $\begin{bmatrix} 2 & 15 \\ 2 & 24 \end{bmatrix}$

18. (a) 19. (c) 20. (b) 21. (b) 22. (a)

1.6 PERMUTAITON AND COMBINATION

Before knowing about concept of permutation and combination first we must be familiar with the term factorial.

Factorial : Factorial of a positive integer 'n' is defined as :

$$n! = n \times (n-1) \times (n-2) \ldots 3 \times 2 \times 1$$

where symbol of factorial is ! or $\lfloor\,\,$. For example

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

Note $0! = 1$

Example 76. Evaluate

(i) $6!$ (ii) $3! + 2!$ (iii) $\frac{5!3!}{2!}$ (iv) $5! + 4!$

Sol.

(i) $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$

(ii) $3! + 2! = 3 \times 2 \times 1 + 2 \times 1 = 6 + 2 = 8$

(iii) $\dfrac{5!3!}{2!} = \dfrac{5\times4\times3\times2\times1\times3\times2\times1}{2\times1} = 360$

(iv) $5! - 3! = 5 \times 4 \times 3 \times 2 \times 1 - 3 \times 2 \times 1 = 120 - 6 = 114.$

Example 77. Evaluate (i) $\dfrac{n!}{(n-2)!}$ (ii) $(n - r)!$ when $n = 7, r = 3$.

Sol. $$\frac{n!}{(n-2)!} = \frac{n\times(n-1)\times(n-2)(n-3)...3\times2\times1}{(n-2)\times(n-3)...3\times2\times1}$$

$$= n(n - 1) = n^2 - n$$

(ii) $(n - r)! = (7 - 3)! = 4! = 4 \times 3 \times 2 \times 1 = 24$

Example 78. Evaluate : (i) $\dfrac{8!-6!}{3!}$ (ii) $\dfrac{2!}{4!}+\dfrac{7!}{5!}$

Sol. (i) $$\frac{8\times7\times6!-6!}{3!} = \frac{6![8\times7-1]}{3!}$$

$$\frac{6!\times55}{3!} = \frac{6\times5\times4\times3!\times55}{3!} = 6600$$

(ii) $$\frac{2!}{4!}+\frac{7!}{5!} = \frac{2!}{4\times3\times2!}+\frac{7\times6\times5!}{5!}$$

$$= \frac{1}{12}+\frac{42}{1} = \frac{1+504}{12} = \frac{505}{12}$$

EXERCISE - VII

1. Compute $|\underline{3} + |\underline{6}$.

2. Evaluate $|\underline{n-r}$ where $n = 8, r = 4$.

3. Evaluate $\frac{10!}{8!3!}$.

4. Evaluate $\frac{7!-5!}{3!}$.

5. Evaluate product 3!.4!.7! and prove that 3! + 4! ≠ 7!.

6. Solve the equation (n + 1)! = 12 n!

ANSWERS

1. 726 2. 24 3. 15 4. 820 5. 725,760 6. n = 3

Permutation : It is the number of arrangements of 'n' different things taken 'r' at a time and is calculated by the formula,

$$^{n}P_{r}=\frac{n!}{(n-r)!}$$

Example 79. Evaluate $^{7}P_{4}$.

Sol. We know $^{n}P_{r}=\frac{n!}{(n-r)!}$

Put n = 7 and r = 4, we get

$$^{7}P_{4}=\frac{7!}{(7-4)!}=\frac{7!}{3!}=\frac{7\times6\times5\times4\times3\times2\times1}{3\times2\times1}=840$$

Example 80. Evaluate (i) $^{6}P_{6}$ (ii) $^{4}P_{1}$

Sol. (i) $^{6}P_{6}=\frac{6!}{(6-6)!}=\frac{6!}{0!}=\frac{6\times5\times4\times3\times2\times1}{1}=720$.

(ii) $^{4}P_{1}=\frac{4!}{(4-1)!}=\frac{4!}{3!}=\frac{4\times3!}{3!}=4$.

Combination : It is the grouping of 'n' different things taken 'r' at a time and is calculated by the formula.

$$^{n}c_{r} = \frac{n!}{(n-r)!r!}$$

Example 81. Evaluate : (i) $^{9}c_{5}$ (ii) $^{n}c_{0}$ (iii) $^{n}c_{n}$

Sol. (i) As $^{n}c_{r} = \frac{n!}{(n-r)!r!}$

Put n = 9, r = 5, we get

$$^{9}c_{5} = \frac{9!}{(9-5)!5!} = \frac{9!}{4!5!} = \frac{9\times8\times7\times6\times5!}{4\times3\times2\times1\times5!} = 126$$

(ii) $$^{n}c_{0} = \frac{n!}{(n-0)!0!} = \frac{n!}{n!\,0!} = 1 \quad \text{as } 0! = 1$$

(iii) $$^{n}c_{n} = \frac{n!}{(n-n)!n!} = \frac{n!}{0!n!} = 1$$

EXERCISE -VIII

1. Define permutation and combination with examples.

2. if n = 10, r = 4 then find value $\frac{n!}{(n-r)!}$.

3. Evaluate (i) $^{10}P_{2}$ (ii) $^{5}P_{5}$ (iii) $^{8}c_{3}$ (iv) $^{6}c_{0}$

4. Find n if $^{n}p_{2} = 20$.

5. Find the value of $^{10}c_{3} + ^{10}c_{4}$.

6. If $11.\, ^{n}P_{4} = 20.\, ^{n-2}P_{4}$ then the value of n is
   (a) 38 (b) 20 (c) 16 (d) None of these

7 - The value of 1.3.5……(2n -1) . 2^{n} equals
   (a) $\frac{(2n)!}{2^{n}}$ (b) $\frac{n!}{2^{n}}$ (c) $\frac{(2n)!}{n!}$ (d) None of these

8 - If $^{n}C_{r-1} = 36$, $^{n}C_{r} = 84$, $^{n}C_{r+1} = 126$ then r is equal to
   (a) 3 (b) 2 (c) 1 (d) None of these

9 - The value of 1 . 1! + 2 . 2! + 3 . 3! + 4 . 4! is
   (a) 118 (b) 119 (c) 120 (d) None of these

ANSWERS

2. 5040 3. (i) 90 (ii) 1 (iii) 56 (iv) 1 4. 5 5. 330 6. (c) 7. (c)

8. (a) 9. (b)

1.7 BINOMIAL THEOREM

Binomial Theorem for Positive Integer: If n is any positive integer, then

$(x+a)^n = {}^nc_0 x^{n-0} a^0 + {}^nc_1 x^{n-1} a^1 + {}^nc_2 x^{n-2} a^2 + {}^nc_n x^{n-n} a^n$ is called Binomial expansion, where ${}^nc_0, {}^nc_1, {}^nc_2, ..., {}^nc_n$ are called Binomial co-efficients.

Features of Binomial Theorem :

(i) The number of terms in Binomial expansion is one more than power of Binomial expression.

(ii) In Binomial expansion the sum of indices of x and a is equal to n.

(iii) The value of Binomial co-efficient, equidistant from both ends is always same.

Application in Real Life :

In real life Binomial theorem is widely used in modern world areas such a computing, i.e. Binomial Theorem has been very useful such as in distribution of IP addresses. Similarly in nation's economic prediction, architecture industry in design of infrastructure etc.

Example 82. How may terms are there in binomial expansion of $(a + b)^7$.

Sol. The number of terms in binomial expansion of $(a + b)^7$ is (n + 1) where n = 7. So total number of terms in expansion = 8.

Example 83. Which of the binomial co–efficients have same value in $(x + a)^7$ ${}^7c_0, {}^7c_1, {}^7c_2, {}^7c_3, {}^7c_4, {}^7c_5, {}^7c_6, {}^7c_7$.

Sol. ${}^7c_0 = {}^7c_7 = 1,$ ${}^7c_1 = {}^7c_6 = 7$

$$^7c_2 = {}^7c_5 = 21 \qquad {}^7c_3 = {}^7c_4 = 35$$

Example 84. Expand $(x + y)^7$ binomially.

Sol. $(x+y)^7 = {}^7c_0x^7y^0 + {}^7c_1x^6y^1 + {}^7c_2x^5y^2 + {}^7c_3x^4y^3 + {}^7c_4x^3y^4$

$$+{}^7c_5x^2y^5 + {}^7c_6x^1y^6 + {}^7c_7x^0y^7$$

(1)

as $\quad {}^7c_0 = {}^7c_7 = 1, \qquad {}^7c_1 = {}^7c_6 = 7$

$$^7c_2 = {}^7c_5 = 21 \qquad {}^7c_3 = {}^7c_4 = 35$$

so equation (1) becomes

$$(x+y)^7 = x^7 + 7x^6y^1 + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

EXERCISE - IX

1. State Binomial Theorem for n as a positive integer.
2. Write the number of terms in expansion of $(x + y)^{10}$.
3. Which of Binomial co–efficients in binomial expansion of $(a + b)^8$ have same value. Also evaluate.
4. Expand $(x + 2y)^5$ using Binomial theorem.
5. Expand $\left(x+\frac{1}{x}\right)^6$ using Binomial Theorem.
6. Expand $(2x - 3y)^4$ using Binomial Theorem.
7. Expand $(a^2 + b^3)^4$ using Binomial Theorem.

ANSWERS

2. 11   3. ${}^8c_0 = {}^8c_8 = 1, {}^8c_1 = {}^8c_7 = 8, \ {}^8c_2 = {}^8c_6 = 28, \ {}^8c_3 = {}^8c_5 = 56, {}^8c_4 = 70$

4. $(x+2y)^5 = x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$

5. $\left(x + \frac{1}{x}\right)^6 = x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$

6. $(2x-3y)^4 = 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$

7. $(a^2+b^3)^4 = a^8 + 4a^6b^3 + 6a^4b^6 + 4a^2b^9 + b^{12}$

General Term of Binomial Expression $(x + a)^n$ for positive integer 'n'

$$T_{r+1} = {}^nc_r x^{n-r} a^r \qquad \text{where } 0 \le r \le n$$

Generally we use above formula when we have to find a particular term.

Example 85. Find the 5^{th} term in expansion of $(x - 2y)^7$.

Sol. Compare $(x - 2y)^7$ with $(x + a)^n$

$\Rightarrow$ $x = x$, $a = -2y$, $n = 7$

$T_5 = T_{r+1}$ $\Rightarrow$ $r + 1 = 5$, $r = 4$

Using $T_{r+1} = {}^nc_r x^{n-r} a^r$

Put all values

$$T_{4+1} = {}^7c_4 x^{7-4} (-2y)^4$$

$$= \frac{7!}{3!4!} x^3 16y^4$$

$$= 560\, x^3\, y^4$$

Note : p^{th} term from end in expansion of $(x + a)^n$ is $(n - p + 2)^{th}$ term from starting.

To find Middle term in $(x + a)^n$ when n is positive integer

(a) when n is even positive integer,

$$\text{Middle term} = T_{\frac{n}{2}+1}$$

(b) When n is odd positive integer

$$\text{Middle term} = T_{\frac{n+1}{2}} \text{ and } T_{\frac{n+1}{2}+1}$$

Example 86. In Binomial expression (i) $(x + y)^{10}$ (ii) $(x + y)^{11}$. How many middle terms are there.

Sol. (i) In $(x + y)^{10}$

This binomial expression has only one Middle Term

i.e. $T_{\frac{10}{2}+1} = T_6$

(ii) In $(x + y)^{11}$

This Binomial expression has two middle terms

i.e. $T_{\frac{11+1}{2}}$ and $T_{\frac{11+1}{2}+1}$

or T_6 and T_7.

Example 87. The 3^{rd} term from end in binomial expression of $(x - 2y)^7$ is _____ term from starting.

Sol. Using formula $(n - p + 2)^{th}$ term.

Here n = 7, p = 3.

So 3^{rd} term end in binomial expansion of $(x -2y)^7$ is $(7 - 3 + 2)^{th}$ term from starting, i.e. 6^{th} from starting.

Example 88. Find the middle term in expansion of $\left(\frac{x}{y}+\frac{y}{x}\right)^{12}$.

Sol. Middle term = $T_{\frac{12}{2}+1} = T_7$

Compare $\left(\frac{x}{y}+\frac{y}{x}\right)^{12}$ with $(x + a)^n$

$X = \frac{x}{y}$, $A = \frac{y}{x}$, $N = 12$

Using $T_{R+1} = {}^nC_r x^{n-r} a^r$

$T_{R+1} = T_7 \quad \Rightarrow \quad R = 6$

$T_{6+1} = {}^{12}C_6 \left(\frac{x}{y}\right)^{12-6} \left(\frac{y}{x}\right)^6$

$= 924 \left(\frac{x}{y}\right)^6 \left(\frac{y}{x}\right)^6$

$T_{6+1} = 924$

Example 89. Find 5^{th} term from end in expansion of $\left(\frac{x^2}{2}-\frac{2}{x^3}\right)^9$.

Sol. Compare $\left(\frac{x^2}{2}-\frac{2}{x^3}\right)^9$ with $(x + a)^n$

$x = \frac{x^2}{2}$, $a = -\frac{2}{x^3}$, $n = 9$

5^{th} term from end is $(n - p + 2)^{th}$ term from staring.

i.e. $(9 - 5 + 2)^{th}$ term from starting.

6^{th} term from staring.

Using $T_{R+1} = {}^nC_r x^{n-r} a^r$

$T_{R+1} = T_6 \quad \Rightarrow \quad r + 1 = 6, \; r = 5.$

So, $$T_{5+1} = {}^9C_5 \left(\frac{x^2}{2}\right)^{9-5} \left(-\frac{2}{x^3}\right)^5$$

$$= -126\frac{x^8}{16} \times \frac{32}{x^{15}}$$

$$T_{5+1} = \frac{-252}{x^7}$$

EXERCISE - X

1. Find 4th term of $\left(x + \frac{1}{x}\right)^7$ Binomially.

2. Find the 4th term of $\left(\frac{4x}{7} - y^2\right)^5$ Binomially.

3. Find middle term of $(x - 2y)^7$ using Binomial theorem.

4. Find 5th term in Binomial expansion $\left(x^2 + \frac{1}{x}\right)^7$.

5. Find 3rd term from end in Binomial expansion $(2x - 3)^6$.

6. Find Middle Term in Binomial expansion $\left(\frac{2}{x} - \frac{x}{2}\right)^5$.

7. In Binomial expansion of $\left(\frac{4x}{7} - y^2\right)^5$. Find 4th term.

8. The total number of terms in the expansion of $(x+y)^{100}$ is
   (a) 100 (b) 200 (c) 101 (d) None of these

9. The coefficient of x^4 in $\left(\frac{x}{2} - \frac{3}{x^2}\right)^2$ is
   (a) $\frac{504}{259}$ (b) $\frac{405}{256}$ (c) $\frac{450}{263}$ (d) None of these

10. The constant term in the expansion of $\left(x - \frac{1}{x}\right)^{10}$ is
   (a) 152 (b) -152 (c) 252 (d) -252

11 - In the expansion of $(3x + 2)^4$, the coefficient of middle term is
(a) 95 (b) 64 (c) 236 (d) 216

ANSWERS

1. $35x$ 2. $-\frac{160}{49}x^2y^6$ 3. (i) $-280x^4y^3$ (ii) $560x^3y^4$ 4. $35x^2$

5. $4860x^2$ 6. (i) $\frac{20}{x}$ (ii) $-5x$ 7. $-\frac{160}{49}x^2y^6$ 8. (c) 9. (a)

10. (d) 11. (d)

* * * * *

UNIT - 2

TRIGONOMETRY

Learning Objectives

- To understand angle, angle measurementsand theirconversions.
- To define T-ratios of standard and allied angles, sum, difference and product formulae.
- To apply trigonometric formulae in solving engineering problems.

Introduction: The word trigonometry is derived from two Greek words : trigono meaning ‘a triangle’ and metron meaning ‘to measure’. Thus literally trigonometry means ‘measurement of triangles’. In early stages of development of trigonometry, its scope lied in the measurement of sides and angles of triangles and the relationship between them. Though still trigonometry is largely used in that sense but of late it is also used in many other areas such as the science of seismology, designing electric circuits and many more areas.

2.1CONCEPT OF ANGLE

Definition : According to Euclid ‘an angle is the inclination of a line to another line’. An angle may be of any magnitude and it may be positive or negative.

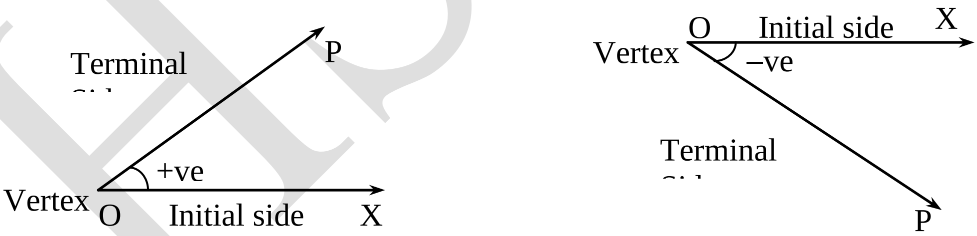

Fig. 2.1

Angle in any quadrant

Two mutually perpendicular straight lines XOX' and YOY' divide the plan of paper into four parts XOY, YOX', X'OY and Y'OX' which are called I, II, III, IV quadrants respectively.

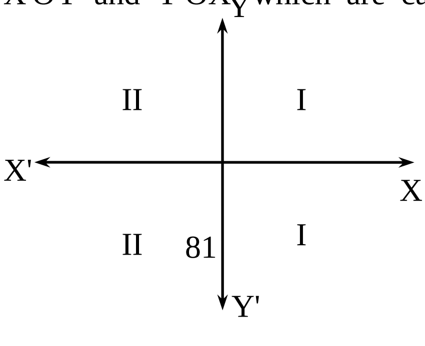

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Fig. 2.2

Measurement of Angle

Sometimes different units are used to measure the same quantity. For example, time is measured in hours, minutes and seconds. In the same manner, we shall now describe three most commonly used units of measurement of an angle.

(i) Sexagesimal OR the English system

(ii) Centesimal OR the French system

(iii) Circular measure system.

Sexagesimal System (Degree measure)

In this system the unit of measurement is a degree. If the rotation from the initial side to terminal side is $\left(\frac{1}{360}\right)^{th}$ of revolution, then the angle is said to have a measure of one degree and is written as 1°.

A degree is further divided into minutes and a minute is divided into second $\left(\frac{1}{60}\right)^{th}$ of a degree is called a minute and $\left(\frac{1}{60}\right)^{th}$ of a minute is called a second. We can write as :

1 right angle = 90 degrees (written as 90°)

1 degree (1°) = 60 minutes (written as 60′)

1 minute (1′) = 60 seconds (written as 60")

Centesimal System : (Grade Measure) In this system a right angle is divided into 100 equal parts, each part called a grade. Each grade is further sub divided into 100 equal parts, called a minute and each minute is again divided into 100 equal parts, called a second. Thus, we have

1 right angle = 100 grades (written as 100^g)

1 grade (1^g) = 100 minutes (written as 100')

1 minute (1') = 100 second (written as 100")

Circular System (Radian measure) : In this system the unit of measurement is radian. A radian is the measure of an angle whose vertex is the centre of a circle and which cuts off an arc equal to the radius of the circle from its circumference:

One radian is shown below:

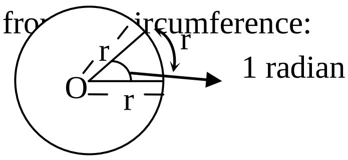


Fig 2.3

Thus a radian is an angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle. One radius is denoted by as 1^c.

$$\pi^c = 180^\circ \qquad \text{OR} \qquad \frac{\pi^c}{2} = 90^\circ$$

We know that the circumferences of a circle of radius r is $2\pi r$. Thus one complete revolution of the initial side subtends an angle of $\frac{2\pi r}{r}$ i.e., 2π radian.

Relation between three systems of an angle measurement

$$90^\circ = 100^g = \frac{\pi^c}{2}$$

OR

$$180^\circ = 200^g = \pi^c (\pi \text{ radius}) \qquad \left[\because \pi = \frac{22}{7}\right]$$

Example 1. Write the following angles in circular measure

(i) 75° (ii) 140°

Sol.: (i) We know that

$$90^0 = \frac{\pi^c}{2}$$

$$1^0 = \frac{\pi^c}{2} x \frac{1}{90}$$

$$75^0 = \frac{\pi^c}{2} x \frac{1}{90} x\ 75 = \frac{5\pi^c}{12}$$

(ii) $$90^0 = \frac{\pi^c}{2}$$

$$1^0 = \frac{\pi^c}{2} x \frac{1}{90}$$

$$140^0 = \frac{\pi^c}{2} x \frac{1}{90} x\, 140 = \frac{7\pi^c}{9}$$

Example 2. Find the centesimal measure of the angle whose radian measure are :

(i) $\frac{4\pi^c}{5}$ (ii) $\frac{3\pi^c}{10}$

Sol.(i) We know that

$$100^g = \frac{\pi^c}{2}$$

$$\frac{\pi^c}{2} = 100^g$$

$$\frac{\pi^c}{2} = 100$$

$$\pi = 200$$

$$\frac{4\pi}{5} = \frac{200}{5} \times 4 = 160^g$$

(ii) $$\frac{\pi^c}{2} = 100^g$$

$$\frac{\pi^c}{2} = 100$$

$$\pi^c = 200$$

$$\frac{3\pi^c}{10} = 200\ x\ \frac{3}{10} = 60^g$$

Example 3. Write the following angles in sexagesimal measure whose radians measures are:

(i) $\frac{\pi^c}{5}$ (ii) $\frac{\pi^c}{6}$

Sol : (i) We know that

$$90^\circ = \frac{\pi^c}{2}$$

$$\pi^c = 180^\circ$$

$$\frac{\pi^c}{5} = 180^0 x \frac{1}{5} = 36^0$$

(ii) We know that

$$\pi^c = 180°$$

$$\frac{\pi^c}{6} = 180^0 x \frac{1}{6} = 30^0$$

EXERCISE- I

1. Express in radians the followings angles

   (i) 45° (ii) 530° (iii) 40° 20'

2. Find the degree measures corresponding to the following radians measures

   (i) $\left(\frac{\pi}{8}\right)^c$ (ii) $\left(\frac{7\pi}{12}\right)^c$ (iii) $\left(\frac{3\pi}{4}\right)^c$

ANSWERS

1. (i) $\frac{\pi}{4}$ radians (ii) $\frac{53\pi}{18}$ radians (iii) $\frac{121\pi}{540}$ radians

2. (i) 22°30' (ii) 105° (iii) 42° 57' 17''

2.2 TRIGONOMETRIC RATIOS OF ANGLES

Trigonometric ratios are used to find the remaining sides and angles of triangles, when some of its sides and angles are given. This problem is solved by using some ratios of sides of a triangle with respect to its acute angles. These ratios of acute angles are called trigonometric ratios.

Sign of fundamental lines

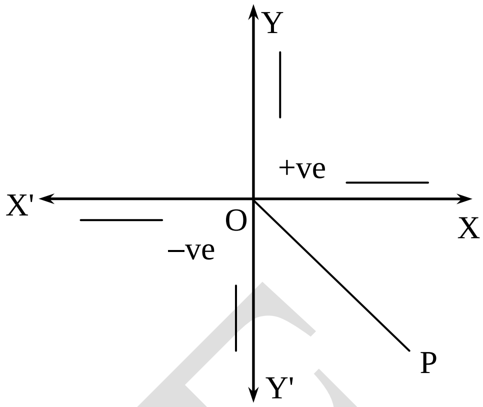
Let XOX' and YOY' be any two mutually perpendicular lines intersecting at O and dividing the plane into four parts

OX (towards the right) as +ve

OX' (towards the left) as –ve

OY (upwards) as +ve

OY'(downwards) as –ve



The revolving line OP is +ve in all its position.

Fig 2.4

Trigonometric Ratios: Let a revolving line OP starting from OX, trace an angle $\angle XOP = \theta$ where θ may be in any quadrant. From P draw perpendicular on XOX'

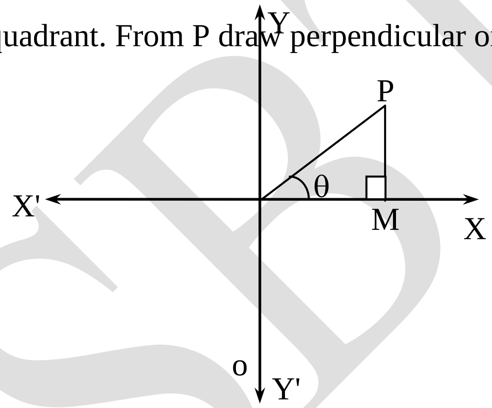


Fig. 2.5

So, in a right angled triangle, ratios are,

(i) $\frac{MP}{OP}\,\frac{(\text{Perpendicular})}{\text{Hypotenuse}}$ is called the sine of angle θ and written as $\sin\theta$.

(ii) $\frac{OM}{OP}\left(\frac{\text{Base}}{\text{Hypotenuse}}\right)$ is called the cosine of angle θ and written as $\cos\theta$.

(iii) $\frac{MP}{OM}\left(\frac{\text{Perpendicular}}{\text{Base}}\right)$ is called the tangent of angle θ and written as $\tan\theta$.

(iv) $\frac{OM}{MP}\left(\frac{\text{Base}}{\text{Perpendicular}}\right)$ is called cotangent of angle θ and written as $\cot\theta$.

(v) $\frac{OP}{OM}\left(\frac{\text{Hypotenuse}}{\text{Base}}\right)$ is called secant of angle θ and written as sec θ.

(vi) $\frac{OP}{MP}\left(\frac{\text{Hypotenus}}{\text{Perpendicular}}\right)$ is called cosecant of angle θ and written as cosec θ.

Relation between trigonometric ratios

(i) $\tan\theta = \frac{\sin\theta}{\cos\theta}$

(ii) $\sin^2\theta + \cos^2\theta = 1$

(iii) $\sec^2\theta - \tan^2\theta = 1$

(iv) $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

Signs of Trigonometric Ratios: The sign of various t-ratios in different quadrants are

(i) In first quadrant all the six t-ratios are positive.

(ii) In second quadrant only sin θ and cosec θ are positive and remaining t-ratios are negative.

(iii) In third quadrant only tan θ and cot θ are positive and remaining t-ratios are negative.

(iv) In fourth quadrant cosθ and sec θ are positive and remaining t-ratios are negative.

The revolving line OP is always positive.

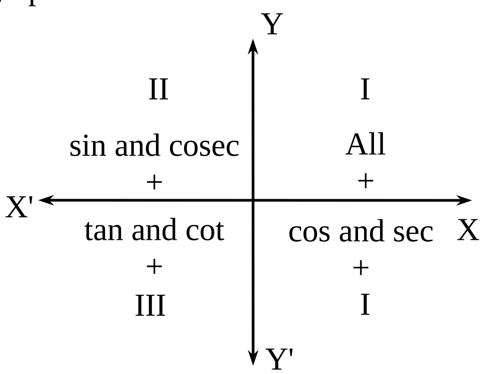


Fig. 2.6

Example4. In a ΔABC, right angle at A, if AB = 12, AC = 5 and BC = 13. Find the value of sin B, cos B and tan B.

Sol. In right angled ΔABC ;

Base = AB = 12,

Perpendicular = AC = 5

Hypotenus = BC = 13

$$\sin B = \frac{AC}{BC} = \frac{5}{13}$$

$$\cos B = \frac{AB}{BC} = \frac{12}{13}$$

$$\tan B = \frac{AC}{AB} = \frac{5}{12}$$

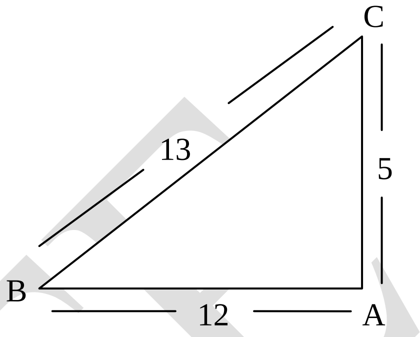


Fig. 2.7

Example 5. In a ΔABC, right angled at B if AB = 4, BC = 3, find the value of sin A and cos A.

Sol : We know by Pythagoras theorem

$$AC^2 = AB^2 + BC^2$$

$$AC^2 = 4^2 + 3^2$$

$$= 16 + 9 = 25$$

$$AC^2 = (5)^2 \qquad \therefore AC = 5$$

$$\therefore \qquad \sin A = \frac{BC}{AC} = \frac{3}{5}$$

$$\cos A = \frac{AB}{AC} = \frac{4}{5}.$$

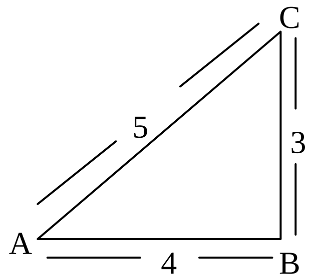


Fig. 2.8

Example 6. If sin A = $\frac{3}{5}$ find the value of cos A and tan A.

Sol : We know that $\sin A = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}} = \dfrac{3}{5}$.

By Pythagoras theorem

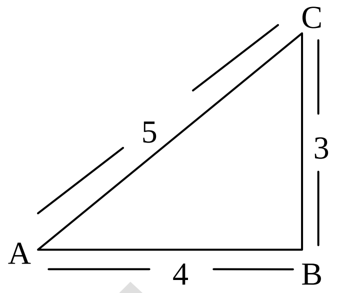


Fig. 2.9

$AC^2 = AB^2 + BC^2$

$5^2 = AB^2 + 3^2$

$25 = AB^2 + 9$

$AB^2 = 25 - 9 = 16$

$AB = 4$

$\cos A = \dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{4}{5}$

$\tan A = \dfrac{\text{Perpendicular}}{\text{Base}} = \dfrac{3}{4}$

Example7. If $\operatorname{cosec} A = \sqrt{10}$. Find the values of sin A, cos A.

Sol : We have $\operatorname{cosec} A = \dfrac{\text{Hypotenuse}}{\text{Perpendicular}} = \dfrac{\sqrt{10}}{1}$

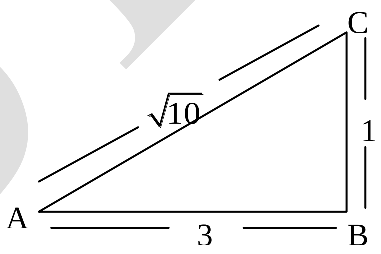


Fig. 2.10

In a right angled triangle ABC.By Pythagoras theorem.

$AC^2 = AB^2 + BC^2$

$(\sqrt{10})^2 = AB^2 + (1)^2$

$10 = AB^2 + 1$

$AB^2 = 9 \qquad \therefore AB = 3$

$\therefore \qquad \sin A = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}} = \dfrac{1}{\sqrt{10}}$

$\cos A = \dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{3}{\sqrt{10}}$

EXERCISE-II

1. In a right triangle ABC, right angle at B, if sin A = $\frac{3}{5}$, find the value of cos A and tan A.

2. In a ΔABC, right angle at B, if AB = 12, BC = 5, find sin A and tan A.

3. In a ΔABC, right angled at B, AB = 24cm, BC 7cm. Find the value of sin A, cos A

4. If tan θ= $\frac{3}{5}$, find the value of $\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$.

5. If 3 tan θ = 4, find the value of $\frac{4\cos\theta-\sin\theta}{2\cos\theta+\sin\theta}$.

6. If cot θ= $\frac{7}{8}$. Find the value of $\frac{(1+\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(1-\cos\theta)}$.

ANSWERS

1. $\cos A=\frac{4}{5}, \tan A=\frac{3}{4}$ 2. $\sin A=\frac{5}{13}, \tan A=\frac{5}{12}$ 3. $\frac{7}{25}, \frac{24}{25}$

4. $\frac{8}{2}=4$ 5. $\frac{4}{5}$ 6. $\frac{49}{64}$.

T-Ratios of Standard Angles

The angles 0°, 30°, 60°, 90°, 180°, 270° and 360° are called standard angles. The value of 0°, 30°, 45°, 60° and 90° can be remembered easily with the help of following table:

Table 2.1

Angleθ	0°	30°	45°	60°	90°
$\sin\theta$	$\sqrt{\frac{0}{4}}=0$	$\sqrt{\frac{1}{4}}=\frac{1}{2}$	$\sqrt{\frac{2}{4}}=\frac{1}{\sqrt{2}}$	$\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$	$\sqrt{\frac{4}{4}}=1$
$\cos\theta$	$\sqrt{\frac{4}{4}}=1$	$\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$	$\sqrt{\frac{2}{4}}=\frac{1}{\sqrt{2}}$	$\sqrt{\frac{1}{4}}=\frac{1}{2}$	$\sqrt{\frac{0}{4}}=0$
$\tan\theta$	$\sqrt{\frac{0}{4}}=0$	$\sqrt{\frac{1}{3}}=\frac{1}{\sqrt{3}}$	$\sqrt{\frac{2}{2}}=1$	$\sqrt{\frac{3}{1}}=\sqrt{3}$	$\sqrt{\frac{4}{0}}=0$

T-ratios of Allied angles :

Allied angles : Two anglesare saidto be allied angles when their sum or differences is either zero or a multiple of 90°.

Complimentary angles : Two angles whose sum is 90° are called complement of each other. The angle θ and 90°–θ are complementary of each other.

Supplementary angles : Two angles whose sum is 180° are called supplementary of each other. The angles θ and 180°–θ are supplementary of each other.

The value of –θ, 90 ±θ, 180 ±θ, 270 ±θ and 360 ±θ can be remember easily by the following table.

Table 2.2

Angle	**Sin**	**Cos**	**Tan**	**Remarks**
–θ	sin(–θ) = –sin θ	cos(–θ) = cosθ	tan (–θ) = tanθ	
90 –θ	cosθ	sin θ	cot θ	Co–formulae apply
90 + θ	cosθ	–sin θ	–cot θ	Co–formulae apply
180 –θ	sin θ	–cosθ	– tan θ	
180 + θ	–sin θ	–cosθ	tan θ	
270 –θ	–cosθ	– sin θ	cot θ	Co–formulae apply
270 + θ	–cosθ	sin θ	–cot θ	Co–formulae apply
360 –θ	–sin θ	cosθ	–tan θ	
360 + θ	sin θ	cosθ	tan θ	

Example 8. Find the value of:

(i) sin 135° (ii) sin 300°.

Sol : (i) sin 135° = sin (90° + 45°) $\because$ sin(90° + **θ**) = cos**θ**

$$= \cos 45° = \frac{1}{\sqrt{2}}$$

OR

sin 135° = sin (180°– 45°)

$$= \sin 45° = \frac{1}{\sqrt{2}}$$

(ii) sin 300° = sin (360°– 60°) $\because$ sin(360°–**θ**) = –sin **θ**

$$= -\sin 60° = \frac{-\sqrt{3}}{2}$$

Example9. Evaluate

(i) tan 120° (ii) sin 150° (iii) cos 300° (iv) cot 225°

(v) sin(–690°)

Sol : (i) tan 120° = tan(90° + 30°) = –cot 30° = $-\sqrt{3}$

(ii) sin 150° = sin(180°– 30°) = sin 30° = $\frac{1}{2}$

(iii) cos 300° = cos(360°– 60°) = cos 60° = $\frac{1}{2}$

(iv) cot 225° = cot (180° + 45°) = cot 45° = 1

(v) sin (–690°) = –sin 690° = –sin (7 × 90° + 60°)

= cos 60° = + cos 60° = $\frac{1}{2}$

Example 10. Evaluate

(i) cos (–750°) (ii) sin (–240°) (iii) sin 765° (iv) cos 1050°

(v) tan (–1575°)

Sol : (i) cos (–750°) = +cos 750°

= cos (2 × 360° + 30°)

$$= \cos 30° = \frac{\sqrt{3}}{2}$$

(ii) $\sin(-240°) = -\sin 240°$

$$= -\sin(180° + 60°)$$

$$= -\sin 60° = -\frac{\sqrt{3}}{2}$$

(iii) $\sin 765° = \sin(2 \times 360° + 45°)$

$$= \sin 45° = \frac{1}{\sqrt{2}}$$

(iv) $\cos(1050°) = \cos(3 \times 360° - 30°)$

$$= \cos(-30°) = \cos 30° = \frac{\sqrt{3}}{2}$$

(v) $\tan(-1575°) = -\tan 1575°$

$$= -\tan(4 \times 360° + 135°)$$

$$= -\tan 135°$$

$$= -\tan(180° - 45°)$$

$$= \tan 45° = 1$$

Ex. 11. Evaluate the following :(i) $\dfrac{\cos 37°}{\sin 53°}$ (ii) $\sin 39° - \cos 51°$

Sol :(i) $\dfrac{\cos 37°}{\sin 53°} = \dfrac{\cos(90° - 53°)}{\sin 53°} = \dfrac{\sin 53°}{\sin 53°} = 1$

(ii) $\sin 39° - \cos 51°$

$$= \sin(90° - 51°) - \cos 51°$$

$$= \cos 51° - \cos 51° = 0.$$

EXERCISE-III

1. Evaluate the following :

(i) $\dfrac{\sin 41°}{\cos 49°}$ (ii) $\dfrac{\tan 54°}{\cot 36°}$ (iii) $\dfrac{\operatorname{cosec} 32°}{\sec 58°}$

2. Evaluate the following :

(i) cosec25°– sec 65° (ii) cot 34°– tan 56° (iii) $\frac{\sin 36°}{\cos 54°} - \frac{\sin 54°}{\cos 36°}$

3. Find the value of :

(i) sin 300° (ii) $\sin\left(-\frac{5\pi}{3}\right)$

ANSWERS

1. (i) 1 (ii) 1 (iii) 1 2. (i) 0 (ii) 0 (iii) 0

3. (i) $-\frac{\sqrt{3}}{2}$ (ii) $\frac{\sqrt{3}}{2}$

Addition and Subtraction Formulae

Addition Formulae **:**

Let a revolving line starting from OX, trace out an angle ∠XOY = A and let it revolve further to trace an angle ∠YOZ = B. So that ∠XOZ = A + B (Addition of angles A and B).

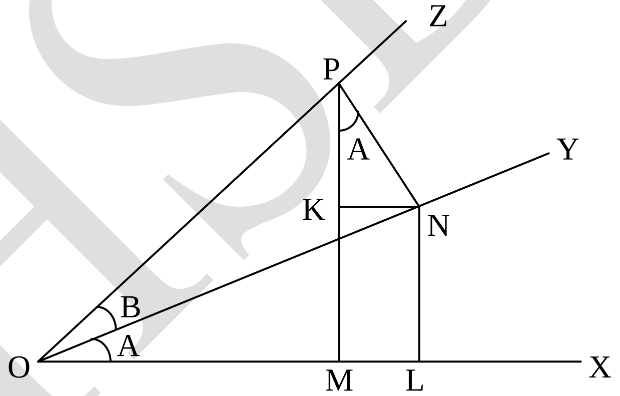


Fig. 2.11

Subtraction formulae **:**

Let a revolving line, starting from OX, trace out an angle ∠XOY = A and let it revolve back to trace an angle ∠YOZ = B. So that ∠XOZ = A –B(Subtraction of Angle A and B)

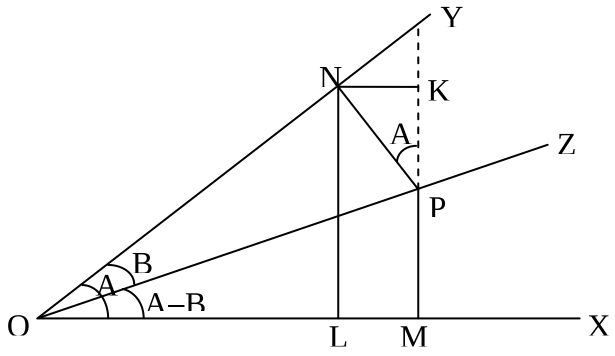


Fig. 2.12

A. Addition Formulae

(1) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

(2) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(3) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(4) $\cot(A + B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$

(5) $\tan(45 + A) = \dfrac{1 + \tan A}{1 - \tan A}$

B. Subtraction Formulae

(1) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

(2) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(3) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(4) $\cot(A - B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

(5) $\tan(45 - A) = \dfrac{1 - \tan A}{1 + \tan A}$

Example 12. Evaluate

(i) $\sin 15°, \cos 15°, \tan 15°$ (ii) $\sin 75°, \cos 75°, \tan 75°$

Sol : (i) (a) sin 15° = sin (45°– 30°)

= sin 45°cos 30°– cos 45° sin 30° [sin(A – B) = sin A cos B – cos A sin B]

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(b) cos 15° = cos (60°– 45°) [cos(A – B) = cosAcosB + sinAsinB]

= cos 60°cos 45° + sin 60° sin 45°

$$= \frac{1}{2}.\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}} = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{1}{4}(\sqrt{2}+\sqrt{6})$$

(c) tan 15° = tan (45°– 30°)

$$= \frac{\tan 45° - \tan 30°}{1+\tan 45° \tan 30°} \qquad \left[\tan(A - B) = \frac{\tan A - \tan B}{1+\tan A \tan B}\right]$$

$$= \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

(ii) (a) sin 75° = sin (45° + 30°)

= sin 45°cos 30° + cos 45° sin 30° $[\because \sin(A+B) = \sin A \cos B + \cos A \sin B]$

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

(b) cos 75° = cos (45° + 30°)

= cos 45°cos 30°– sin 45° sin 30° [cos (A + B) = cos A cos B – sin A sin B]

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(c) tan 75° = tan (45° + 30°)

$$= \frac{\tan 45° + \tan 30°}{1-\tan 45°.\tan 30°} \qquad \left[\tan(A + B) = \frac{\tan A + \tan B}{1-\tan A \tan B}\right]$$

$$= \frac{1+\frac{1}{\sqrt{3}}}{1-1\left(\frac{1}{\sqrt{3}}\right)} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

Example 13. Write down the values of :

(i) $\cos 68°\cos 8° + \sin 68° \sin 8°$ (ii) $\cos 50°\cos 10° - \sin 50° \sin 10°$

Sol :(i) $\cos 68°\cos 8° + \sin 68° \sin 8°$

$= \cos (68° - 8°)$ [$\cos(A - B) = \cos A \cos B + \sin A \sin B$]

$= \cos 60° = \frac{1}{2}$

(ii) $\cos 50°\cos 10° - \sin 50° \sin 10°$

$= \cos (50° + 10°)$ [$\cos(A + B) = \cos A \cos B - \sin A \sin B$]

$= \cos 60° = \frac{1}{2}$

Example 14. Prove that $\frac{\cos 11° + \sin 11°}{\cos 11° - \sin 11°} = \tan 56°$

Sol : L.H.S. = $\frac{\cos 11° + \sin 11°}{\cos 11° - \sin 11°}$

$= \frac{1 + \frac{\sin 11°}{\cos 11°}}{1 - \frac{\sin 11°}{\cos 11°}}$ [Dividing the num. and denom. by cos 11°]

$= \frac{1 + \tan 11°}{1 - \tan 11°} = \tan(45° + 11°)$ $\left[\tan(45^0 + A) = \frac{1 + \tan A}{1 - \tan A}\right]$

$= \tan 56°$ = R.H.S.

Example 15. Prove that $\tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A$.

Sol : We can write, $\tan 3A = \tan (2A + A)$

$\Rightarrow$ $\tan 3A = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$

$\Rightarrow$ $\tan 3A - \tan 3A \tan 2A \tan A = \tan 2A + \tan A$

$\Rightarrow$ $\tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A$

Example 16. If $\tan A = \sqrt{3}$, $\tan B = 2 - \sqrt{3}$, find the value of $\tan (A - B)$.

Sol : Using the formula; $\tan(A - B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\sqrt{3} + 2 - \sqrt{3}}{1 - \sqrt{3}(2 - \sqrt{3})} = \frac{2}{1 - 2\sqrt{3} + 3}$

$$= \frac{2}{4-2\sqrt{3}} = \frac{2}{2(2-\sqrt{3})} = \frac{1}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}$$

$$= \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}$$

$\therefore$ $\tan(A - B) = 2+\sqrt{3}$

Example 17. If A and B are acute angles and $\sin A = \frac{1}{\sqrt{10}}$, $\sin B = \frac{1}{\sqrt{5}}$. Prove that $A+B = \frac{\pi}{4}$.

Sol : Given, $\sin A = \frac{1}{\sqrt{10}}$ and $\sin B = \frac{1}{\sqrt{5}}$

We know, $\cos A = \sqrt{1-\sin^2 A}$ and $\cos B = \sqrt{1-\sin^2 B}$ [$\because$ A and B are acute angles]

$\Rightarrow$ $\cos A = \sqrt{1-\frac{1}{10}}$ and $\cos B = \sqrt{1-\frac{1}{5}}$

$\Rightarrow$ $\cos A = \sqrt{\frac{9}{10}}$ and $\cos B = \sqrt{\frac{4}{5}}$

$\Rightarrow$ $\cos A = \frac{3}{\sqrt{10}}$ and $\cos B = \frac{2}{\sqrt{5}}$

Now cos (A + B) = cos A cos B – sin A sin B

$$= \frac{3}{\sqrt{10}}.\frac{2}{\sqrt{5}} - \frac{1}{\sqrt{10}}.\frac{1}{\sqrt{5}} = \frac{6}{\sqrt{50}} - \frac{1}{\sqrt{50}}$$

$$= \frac{5}{\sqrt{50}} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$$

Hence $A + B = \frac{\pi}{4}$

Example 18. Prove that tan 70° = tan 20° + 2 tan 50°.

Sol : We can write; tan 70° = tan (20° + 50°)

$\Rightarrow$ $\tan 70° = \frac{\tan 20° + \tan 50°}{1 - \tan 20° \tan 50°}$ $\left[\because \tan(A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B}\right]$

$\Rightarrow$ tan 70°– tan 20° tan 50° tan 70° = tan 20° + tan 50°

$\Rightarrow$ tan 70°– tan 20° tan 50° tan (90°– 20°) = tan 20° + tan 50°

$\Rightarrow$ tan 70°– tan 20° tan 50° cot 20° = tan 20° + tan 50°

$\Rightarrow$ tan 70°– tan 50° = tan 20° + tan 50° [$\because$ tan □. cot □ = 1]

$\Rightarrow$ tan 70° = tan 20° + 2 tan 50°

Example 19. Prove that tan 13 A – tan 9A – tan 4A = tan 13A tan 9A tan 4A.

Sol : We can write; tan 13A = tan (9A + 4A)

$$\tan 13A = \frac{\tan 9A + \tan 4A}{1 - \tan 9A \tan 4A} \qquad \left[\because \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}\right]$$

$\Rightarrow$ tan 13A – tan 13A tan 9A tan 4A = tan 9A + tan 4A

$\Rightarrow$ tan 13A – tan 9A – tan 4A = tan 13A tan 9A tan 4A

Example 20. Prove that $\tan 2\theta - \tan\theta = \tan\theta \sec 2\theta$

Sol : L.H.S. = $\tan 2\theta - \tan\theta$

$$= \frac{\sin 2\theta}{\cos 2\theta} - \frac{\sin\theta}{\cos\theta} = \frac{\sin 2\theta \cos\theta - \cos 2\theta \sin\theta}{\cos 2\theta \cos\theta}$$

$$= \frac{\sin(2\theta - \theta)}{\cos 2\theta \cos\theta} = \frac{\sin\theta}{\cos 2\theta \cos\theta} \qquad [\sin(A - B) = \sin A \cos B - \cos A \cos B]$$

$$= \frac{\sin\theta}{\cos\theta \cos 2\theta} = \tan\theta \sec 2\theta.$$

Example 21. Proveby using trigonometric formulae that; tan 65° = tan 25° + 2 tan 40°.

Sol : We can write; 65° = 40° + 25°

$$\tan 65° = \tan(40° + 25°) = \frac{\tan 40° + \tan 25°}{1 - \tan 40° \tan 25°}$$

tan 65°– tan 40° tan 25° tan 65° = tan 40° + tan 25°

tan 65°– tan 40° tan 25° tan (90°– 25°) = tan 40° + tan 25°

tan 65°– tan 40° tan 25° cot 25° = tan 40° + tan 25°

tan 65°– tan 40° = tan 40° + tan 25° [$\because$ tan **θ** cot **θ** = 1]

tan 65° = tan 25° + 2 tan 40°

Hence proved

EXERCISE– IV

1. Evaluate (i) $\sin 105°$ (ii) $\cos 105°$ (iii) $\tan 105°$.

2. Evaluate : (i) $\sin 22°\cos 38° + \cos 22° \sin 38°$ (ii) $\dfrac{\tan 66° + \tan 69°}{1 - \tan 66° \tan 69°}$

3. Prove that :

   (i) $\sin 105° + \cos 105° = \cos 45°$

   (ii) $\cos A = -\dfrac{24}{25}$ and $\cos B = \dfrac{3}{5}$, where $\pi < A < \dfrac{3\pi}{2}; \dfrac{3\pi}{2} < B < 2\pi$; find $\sin(A + B)$ and $\cos(A + B)$

   (iii) If $\tan A = \dfrac{5}{6}$ and $\tan B = \dfrac{1}{11}$. Show that $A + B = \dfrac{\pi}{4}$.

4. Prove that :

   (i) $\tan 28° = \dfrac{\cos 17° - \sin 17°}{\cos 17° + \sin 17°}$

   (ii) $\tan 58° = \dfrac{\cos 13° + \sin 13°}{\cos 13° - \sin 13°}$

5. If $\cos A = \dfrac{1}{7}$ and $\cos B = \dfrac{13}{14}$. Prove that $A - B = 60°$. A and B are acute angles.

6. Prove that :

   (i) $\tan 55° = \tan 35° + 2 \tan 20°$

   (ii) $\tan 50° = \tan 40° + 2 \tan 10°$

   (iii) $2 \tan 70° = \tan 80° - \tan 10°$

7. If $\tan A = \dfrac{1}{2}$ and $\tan B = \dfrac{1}{3}$. Show that $A + B = 45°$. Given that A and B are positive acute angles.

8. If $\tan A = \dfrac{a}{a+1}, \tan B = \dfrac{1}{2a+1}$. Show that $A + B = \dfrac{\pi}{4}$.

9. Prove that $\sqrt{3}\cos 23° - \sin 23° = 2\cos 53°$.

10. If $A + B = \dfrac{\pi}{4}$. Prove that $(1 + \tan A)(1 + \tan B) = 2$.

ANSWERS

1. (i) $\dfrac{\sqrt{3}+1}{2\sqrt{2}}$ (ii) $\dfrac{1-\sqrt{3}}{2\sqrt{2}}$ (iii) $\dfrac{1+\sqrt{3}}{1-\sqrt{3}}$

2. (i) $\dfrac{\sqrt{3}}{2}$ (ii) 1

3. (i) $\dfrac{220}{221}, \dfrac{220}{221}$

Product formulae (Transformation of a Product into a Sum or Difference)

(i) 2 sin A cos B = sin (A + B) + sin (A – B)

(ii) 2 cos A sin B = sin (A + B) – sin (A – B)

(iii) 2 cos A cos B = cos (A + B) + cos (A – B)

(iv) 2 sin A sin B = cos (A – B) –cos (A + B)

Aid to memory

2 sin A cos B = sin (sum) + sin (difference)

2 cos A sin B = sin (sum) – sin (difference)

2 cos A cos B = cos (sum) + cos (difference)

2 sin A sin B = cos (difference) –cos (sum)

Example 22. Express the following as a sum or difference

(i) 2 sin 5x cos 3x (ii) 2 sin 4x sin 3x (iii) 8 cos 8x cos 4x

Sol :(i) 2 sin 5x cos 3x = sin (5x + 3x) + sin (5x – 3x)

= sin 8x + sin 2x

(ii) 2 sin 4x sin 3x = cos (4x – 3x) –cos (4x + 3x) = cos x –cos 7x.

(iii) 8 cos 8x cos 4x = 4[2 cos 8x cos 4x] = 4[cos (8x + 4x) + cos (8x – 4x)]

= 4[cos 12 x + cos 4x]

Example 23. Prove that cos 20°cos 40°cos 60°cos 80°= $\frac{1}{16}$.

Sol : L.H.S. = cos 20°cos 40° $\frac{1}{2}$ cos 80°

$$= \frac{1}{2}\ (\cos 20^\circ \cos 40^\circ)\cos 80^\circ$$

$$= \frac{1}{4}\ (2\cos 20^\circ \cos 40^\circ)\cos 80^\circ$$

$$= \frac{1}{4}\ (\cos 60^\circ + \cos 20^\circ)\cos 80^\circ$$

$$= \frac{1}{4}\left[\frac{1}{2} + \cos 20^{0}\right]\cos 80^{O}$$

$$= \frac{1}{4}\left[\frac{1}{2}\cos 80^\circ + \cos 20^\circ \cos 80^\circ\right]$$

$$= \frac{1}{4}\left[\frac{1}{2}\cos 80^\circ + \frac{1}{2}(2\cos 20^\circ \cos 80^\circ)\right]$$

$$= \frac{1}{8}\ [\cos 80^\circ + \cos 100^\circ + \cos 60^\circ]$$

As cos 100° = cos (180°– 80°) = –cos 80°

$$= \frac{1}{8}\ [\cos 80^\circ - \cos 80^\circ + \frac{1}{2}] = \frac{1}{16} = \text{R.H.S.}$$

Example 24. Prove that,cos 10°cos 50°cos 70°= $\frac{\sqrt{3}}{8}$.

Sol : LHS = $\frac{1}{2}$ [cos 10°(2cos 50°cos 70°)]

$$= \frac{1}{2}\ [\cos 10^\circ(\cos 120 + \cos 20^\circ)]$$

$$= \frac{1}{2}\ [\cos 10^\circ(-\frac{1}{2} + \cos 20^\circ)]$$

$$= -\frac{1}{4}\cos 10^\circ + \frac{1}{4}(2\cos 10^\circ \cos 20^\circ)$$

$$= -\frac{1}{4}\cos 10^\circ + \frac{1}{4}(\cos 30^\circ + \cos 10^\circ)$$

$$= -\frac{1}{4}\cos 10° + \frac{1}{4}\cos 30° + \frac{1}{4}\cos 10°$$

$$= \frac{1}{4}\cos 30° = \frac{1}{4}\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8} = \text{R.H.S}$$

Example 25. Prove that sin 20° sin 40° sin 60° sin 80°= $\frac{3}{16}$.

Sol : L.H.S $= \frac{\sqrt{3}}{2}\sin 20° \sin 40° \sin 80°$ [$\sin 60^{O} = \frac{\sqrt{3}}{2}$]

$$= \frac{1}{2}.\frac{\sqrt{3}}{2}\sin 20°(2\sin 80° \sin 40°)$$

$= \frac{\sqrt{3}}{4}\sin 20°(\cos 40° - \cos 120°)$ ∵ 2sinAsinB=cos(A–B)–cos(A+B)

$$= \frac{\sqrt{3}}{8}(2\sin 20° \cos 40°) + \frac{\sqrt{3}}{8}\sin 20°$$

$$= \frac{\sqrt{3}}{8}(\sin 60° - \sin 20°) + \frac{\sqrt{3}}{8}\sin 20°$$

$$= \frac{\sqrt{3}}{8}\sin 60° - \frac{\sqrt{3}}{8}\sin 20° + \frac{\sqrt{3}}{8}\sin 20°$$

$$= \frac{\sqrt{3}}{8}.\frac{\sqrt{3}}{2} = \frac{3}{16} = \text{R.H.S.}$$

Transformation of a sum or difference into a product formulae

(i) sin C + sin D = 2 sin $\frac{C+D}{2}\cos\frac{C-D}{2}$

(ii) sin C – sin D = $2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(iii) cos C + cos D = $2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(iv) cos C –cos D = $2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$

Example 26. Express the following as product :

(i) sin 14 x + sin 2x (ii) cos 10°–cos 50° (iii) sin 80°– sin 20°

Sol :(i) sin 14x + sin 2x = $2\sin\frac{14x+2x}{2}\cos\frac{14x-2x}{2} = 2\sin 8x\cos 6x$.

(ii) cos 10°–cos 50° = $2\sin\frac{10°+50°}{2}\sin\frac{50°-10°}{2} = 2\sin 30°\sin 20°$

(iii) sin 80°– sin 20° = $2\cos\frac{80°+20°}{2}\sin\frac{80°-20°}{2} = 2\cos 50°\sin 30°$

Example 27. Prove that

(i) $\frac{\sin A+\sin B}{\cos A+\cos B} = \tan\frac{A+B}{2}$, (ii) $\frac{\cos 8x-\cos 5x}{\sin 17x-\sin 3x} = \frac{-\sin 2x}{\cos 10x}$

Sol : (i) L.H.S. = $\frac{\sin A+\sin B}{\cos A+\cos B} = \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}$

$$= \frac{\sin\frac{A+B}{2}}{\cos\frac{A+B}{2}} = \tan\left(\frac{A+B}{2}\right) = \text{R.H.S.}$$

(ii) L.H.S. = $\frac{\cos 9x-\cos 5x}{\sin 17x-\sin 3x} = \frac{2\sin\frac{9x+5x}{2}\sin\frac{5x-9x}{2}}{2\cos\frac{17x+3x}{2}\sin\frac{17x-3x}{2}}$

$$= \frac{-2\sin 7x\sin 2x}{2\cos 10x\sin 7x} = \frac{-\sin 2x}{\cos 10x} = \text{R.H.S.}$$

Example 28. Prove that $\frac{\cos 4x+\cos 3x+\cos 2x}{\sin 4x+\sin 3x+\sin 2x} = \cot 3x$.

Sol : $\frac{\cos 4x+\cos 2x+\cos 3x}{\sin 4x+\sin 2x+\sin 3x}$

$$= \frac{2\cos\frac{4x+2x}{2}\cos\frac{4x-2x}{2}+\cos 3x}{2\sin\frac{4x+2x}{2}\cos\frac{4x-2x}{2}+\sin 3x}$$

$$= \frac{2\cos 3x\cos x+\cos 3x}{2\sin 3x\cos x+\sin 3x}$$

$$= \frac{\cos 3x(2\cos x+1)}{\sin 3x(2\cos x+1)} = \frac{\cos 3x}{\sin 3x} = \cot 3x \text{ .}$$

Example 29. Prove that

(i) $\dfrac{\sin A+\sin B}{\cos A+\cos B}=\tan\dfrac{A+B}{2}$ (ii) $\dfrac{\sin 7A+\sin 3A}{\cos 7A+\cos 3A}=\tan 5A$.

Sol :(i) L.H.S = $\dfrac{\sin A+\sin B}{\cos A+\cos B}$

$$=\frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}=\tan\frac{A+B}{2}=\text{R.H.S.}$$

(ii) L.H.S = $\dfrac{\sin 7A+\sin 3A}{\cos 7A+\cos 3A}$

$$=\frac{2\sin\left(\frac{7A+3A}{2}\right)\cos\left(\frac{7A-3A}{2}\right)}{2\cos\left(\frac{7A+3A}{2}\right)\cos\left(\frac{7A-3A}{2}\right)}\qquad\text{[Using CD formula]}$$

$$=\frac{\sin 5A\cos 2A}{\cos 5A\cos 2A}=\tan 5A=\text{R.H.S.}$$

Example 30. Prove that

(i) $\sin 47° + \cos 77° = \cos 17°$ (ii) $\sin 51° + \cos 81° = \cos 21°$

(iii) $\cos 52° + \cos 68° + \cos 172° = \cos 20° + \cos 100° + \cos 140°$

Sol : L.H.S = $\sin 47° + \cos 77°$

$= \sin 47° + \cos (90°- 13°) = \sin 47° + \sin 13°$

$= 2\sin\dfrac{47°+13°}{2}\cos\dfrac{47°-13°}{2}$ $\left[\because \sin C+\sin D=2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}\right]$

$= 2\sin 30°\cos 17° = \dfrac{2}{2}\cos 17° = \cos 17°$ = RHS $\left[\because \sin 30 = \dfrac{1}{2}\right]$

(ii) L.H.S. = $\sin 51° + \cos 81° = \sin (90°- 39°) + \cos 81°$

$= \cos 39° + \cos 81°$ [$\because \sin (90°-□) = \cos□$]

$= 2\cos\dfrac{81°+39°}{2}\cos\dfrac{81°-39°}{2}$ $\left[\because \cos C+\cos D=2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}\right]$

$= 2\cos 60°\cos 21°$

$= 2\times\dfrac{1}{2}\cos 21° = \cos 21°$ R.H.S. $\left[\because \cos 60°=\dfrac{1}{2}\right]$

L.H.S = cos 52° + cos 68° + cos 172°

$$= 2\cos\frac{52°+68°}{2}\cos\frac{68°-52°}{2}+\cos 172°$$

= 2 cos 60°cos 8° + cos 172°

$= 2 \times \frac{1}{2}$ cos 8° + cos (180°– 8°)

= cos 8°–cos 8° = 0 $[\because \cos(180°-\theta) = -\cos\theta]$

R.H.S = cos 20° + cos 100° + cos 140°

$$= 2\cos\frac{100°+20°}{2}\cos\frac{100°-20°}{2}+\cos 140°$$

= 2 cos 60°cos 40° + cos 140°

$= 2 \times \frac{1}{2}$ cos 40° + cos (180°– 40°)

= cos 40°–cos 40° = 0 $[\because \cos(180°-\theta) = -\cos\theta]$

$\therefore$ L.H.S = R.H.S

Example 31. cos A + cos(120° - A) + cos(120° + A) = 0

Sol: L.H.S = cos A + cos (120°– A) + cos (120° + A)

$$= \cos A + 2\cos\left(\frac{120°+A+120°-A}{2}\right)\cos\left(\frac{120°+A-120°+A}{2}\right)$$

= cos A + 2 cos 120°cos A = cos A + 2$\left(-\frac{1}{2}\right)\cos A$

= cos A –cos A = 0 = R.H.S

EXERCISE- V

1. Express as sum or difference:

   (i) 2 sin 4θ cos 2θ (ii) 2 sin θ cos 3θ

2. Prove that sin 10° sin 30° sin 50° sin 70°= $\frac{1}{16}$.

3. Prove that $\cos 20°\cos 40°\cos 80° = \frac{1}{8}$.

4. Prove that $\sin 10° \sin 50° \sin 60° \sin 70° = \frac{\sqrt{3}}{16}$.

5. Express the following as a product :

   (i) $\sin 7\theta + \sin 3\theta$ (ii) $\cos 5\theta + \cos 3\theta$

   (iii) $\sin 5\theta - \sin \theta$ (iv) $\cos 2\theta - \cos 4\theta$

6. Prove that : (i) $\frac{\cos A - \cos 3A}{\sin 3A - \sin A} = \tan 2A$ (ii) $\frac{\sin 7x + \sin 3x}{\cos 7x + \cos 3x} = \tan 5x$

7. Prove that $\cos 28° - \sin 58° = \sin 2°$.

8. Prove that :

   (i) $\cos 52° = \cos 68° + \cos 172° = 0$

   (ii) $\sin 50° - \sin 70° = \sin 10° = 0$

9. Prove that $\sqrt{3}\cos 13° + \sin 13° = 2\sin 13°$.

10. $\frac{\sin 11A \sin A + \sin 7A \sin 3A}{\cos 11A \sin A + \cos 7A \sin 3A} = \tan 8A$.

ANSWERS

1. (i) $\sin 6\theta + \sin 2\theta$ (ii) $\sin 4\theta - \sin 2\theta$

5. (i) $2\sin 5\theta \cos 2\theta$ (ii) $2\cos 3\theta \sin 2$ (iii) $2\cos 4\theta \cos\theta$ (iv) $2\sin 3\theta \sin\theta$

T–Ratios of Multiple and Submultiple Angles

(i) $\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$.

(iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$.

Remember

(i) sin(any angle) = 2 sin (half angle) cos (half angle)

(ii) cos (any angle) = $\cos^2$ (half angle) – $\sin^2$ (half angle)

= 2 $\cos^2$ (half angle) – 1

= 1 – 2 $\sin^2$ (half angle)

$= \dfrac{1-\tan^2(\text{half angle})}{1+\tan^2(\text{half angle})}$

(iii) tan(any angle) = $\dfrac{2\tan(\text{half angle})}{1-\tan^2(\text{half angle})}$

Remember

(i) $\sin^2$(any angle) = $\dfrac{1-\cos(\text{double the angles})}{2}$

(ii) $\cos^2$ (any angle) = $\dfrac{1+\cos(\text{double the angles})}{2}$

(iii) $\tan^2$ (any angle) = $\dfrac{1-\cos(\text{double the angles})}{1+\cos(\text{double the angles})}$

T–ratios of 3A in terms of those of A

(i) sin 3A = 3 sin A – 4 $\sin^3$ A

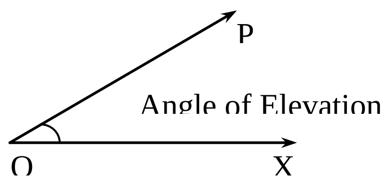
(ii) cos 3A = 4 $\cos^3$ A – 3 cos A

(iii) tan 3A = $\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

2.3 APPLICATIONS OF TRIGONOMETRIC TERMS IN ENGINEERING PROBLEMS

Height and Distance: Trigonometry helps to find the height of objects and the distance between points.

Angle of Elevation : The angle of elevation is for objects that are at a level higher than that of the observer.



$\angle$XOP is called angle of elevation.

Fig. 2.13

Angle of Depression : The angle of depression is for objects that are at a level which is lower than that of the observer.

$\angle$XOP' is called angle of depression.

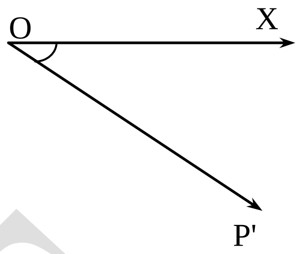


Fig. 2.14

Example 32. A tower is $100\sqrt{3}$ metres high. Find the angle of elevation of its top from a point 100 metres away from its foot.

Sol : Let AB the tower of height $100\sqrt{3}$ m and let C be a point at a distance of 100 metres from the foot of tower. Let θ be the angle of elevation of the top of the lower from point c.

In right angle ΔCAB

$$\frac{AB}{BC} = \tan\theta$$

$$\frac{100\sqrt{3}}{100} = \tan\theta$$

$$\therefore \quad \tan\theta = \sqrt{3} = \tan 60^\circ$$

$$\Rightarrow \quad \theta = 60^\circ,$$

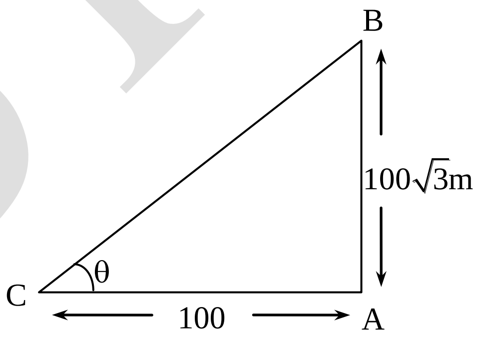
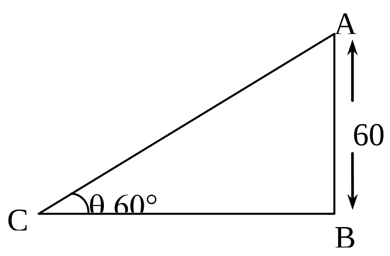


Fig. 2.15

Hence the angle of elevation of the top of the tower from a point 100 metre away from its foot is 60°.

Example 33. A kite is flying at a height of 60 metres above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is 60°. Find the length of the string assuming that there is no slack in the string.

Sol : Let A be the kite and CA be the string attached to the kite such that its one end is tied to



a point C on the ground. The inclination of the string CA with the ground is 60°.

In right angle ΔABC we have

$$\frac{AB}{AC} = \sin\theta$$

$$\frac{AB}{AC} = \sin 60°$$

Fig. 2.16

$$\frac{60}{AC} = \frac{\sqrt{3}}{2} \qquad \text{or} \qquad AC = \frac{120}{\sqrt{3}} = 40\sqrt{3}\text{m}.$$

Hence the length of the string is $40\sqrt{3}$ metres.

Example 34. The string of a kite is 100 metres long and it make and angle of 60° with the horizontal. Find the height of the kite assuming that there is no slack in the string.

Sol : Let CA be the horizontal ground and let B be the position of the kite at a height h above the ground. The AB = h.

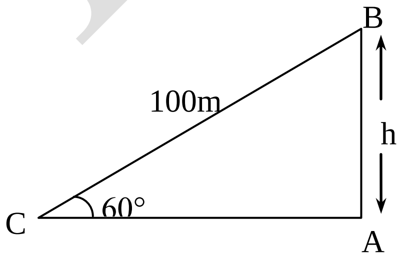


Fig. 2.17

In right angle ΔCAB

$$\frac{AB}{CB} = \sin 60°$$

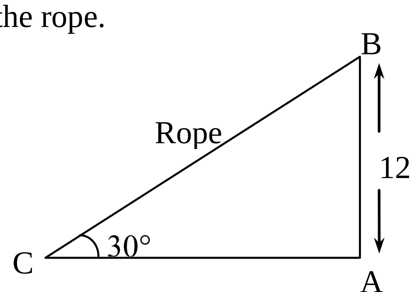
$$\frac{h}{100} = \frac{\sqrt{3}}{2} \qquad \text{or} \qquad h = 100\frac{\sqrt{3}}{2}$$

$\therefore$ $h = 50\sqrt{3}$ metres

Hence the height of the kite is $50\sqrt{3}$ metres.

Example 35. A circus artist is climbing from the ground along a rope stretched from the top of a vertical pole and tied at a ground. The height of the pole is 12m and the angle made by the rope with the ground level is 30°. Calculate the distance covered by the artist in climbing to the top of the pole ?

Sol : Let vertical pole AB of height in metres and CB be the rope.



In right angle ΔCAB

$$\frac{AB}{CB} = \sin\theta = \sin 30°$$

$$\frac{12}{CB} = \frac{1}{2} \qquad \therefore\ CB = 24 \text{ m}$$ Fig. 2.18

Hence the distance covered by the circus artist is 24 m.

Example 36. Two polls of equal height stand on either side of a roadways which is 40 metres wide at a point in the roadway between the polls. The elevation of the tops of the polls are 60° and 30°. Find their height and the position of the point ?

Sol : let AB = 40 metres be the width at the roadway. Let AD = h, BC = h metres be the two polls. Let P be any point on AB at which the ngle of elevation of the tops are 60° and 30°.

Then $\angle APD = 60°$ and $\angle BPC = 30°$

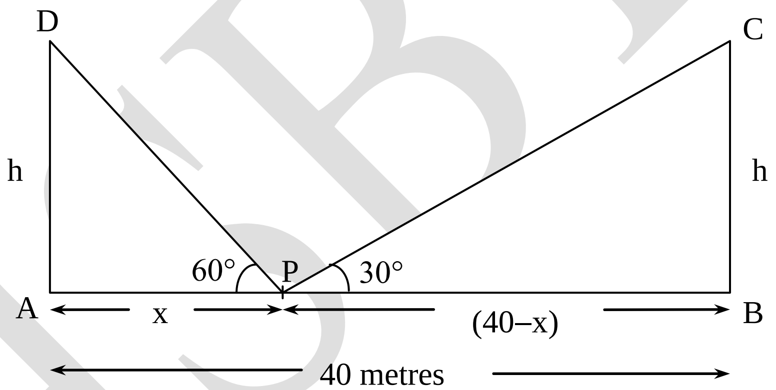


Fig. 2.19

Let AP = x $\qquad \therefore$ PB = 40 – x

Now from right angle Δ PBC

$$\frac{BC}{BP} = \tan 30° \qquad \text{or} \qquad \frac{h}{40 - x} = \frac{1}{\sqrt{3}}$$

$$\sqrt{3}h = 40 - x \qquad \ldots\ldots \text{(i)}$$

Again from right angle ΔPAD

$$\frac{AD}{PA} = \tan 60° \qquad \text{or} \qquad \frac{h}{x} = \sqrt{3}$$

$\therefore \qquad h = \sqrt{3}x \qquad \ldots . . \text{(ii)}$

Substituting the value of h in eqn**.** (i), we get

$$\sqrt{3} \times \sqrt{3}x = 40 - x \qquad \text{or} \qquad 3x = 40 - x$$

$$4x = 40 \qquad \text{so} \qquad x = 10 \text{ metres.}$$

If x = 10 metres than h = $\sqrt{3}.10$ = 17.32 metres (height of polls)

Hence the point P divides AB in the ratio 1 : 3.

EXERCISE- VI

1. A tower stands vertically on the ground from a point on the ground, 20 m away from the foot of the tower, the angle of elevation of the top of the tower is 60°. What is the height of the tower ?
2. The angle of elevation of a ladder leaning against a wall is 60° and the foot of the ladder is 9.5m away from the wall. Find the length of the ladder.
3. A ladder is placed along a wall of a house such that upper end is touching the top of the wall. The foot of the ladder is 2m away from the wall and ladder is making an angle of 60° with the level of the ground. Find the height of the wall ?
4. A telephone pole is 10 m high. A steel wire tied to tope of the pole is affixed at a point on the ground to keep the pole up right. If the wire makes an angle 45° with the horizontal through the foot of the pole, find the length of the wire.
5. A kite is flying at a height of 75 metres from the ground level, attached to a string inclined at 60° to the horizontal. Find the length of the string to the nearest metre.

6. A vertical tower stands on a horizontal plane a is surmounted by a vertical flag–staff. At a point on the plane 70 metres away from the tower, an observer notices that the angle of elevation of the top and bottom of the flag–staff are respectively 60° and 45°. Find the height of the flag–staff and that of the tower. A circus artist is climbing a 20 metre long rope which is tightly stretched and tied from the top of a vertical pole to the

ground. Find the height of the pole if the angle made by the rope with the ground level is 30°.

7. A person standing on the bank of a river, observer that the angle subtended by a tree on the opposite bank is 60°. When the retreats 20 metres from the bank, he finds the angle to be 30°. Find the height of the tree and the breadth of the river ?

9. The magnitude of a radian is

(a) 60^0 (b) $57^0\ 17'\ 44.8''$ nearly (c) $58^0\ 59'$ (d) None of these

10. Angular measurement of an angle is

(a) The number of degrees in an angle
(b) The number of radians in an angle
(c) The number of grades in an angle
(d) None of these

11. The angle subtended by an arc of 1 meter at the centre of a circle with 3 meter radius is

(a) 60^0 (b) 20^0 (c) $\frac{1}{3}$ (d) 3 radian

12. sin (A + B) . sin (A - B) =

(a) $\sin^2 A - \sin^2 B$ (b) $\cos^2 A - \cos^2 B$
(c) $\sin (A^2 - B^2)$ (d) $\sin^2 A - \cos^2 B$

13. The value of $\sin 60^0 \cos 30^0 + \cos 300^0 \sin 330^0$ is

(a) 1 (b) -1 (c) 0 (d) None of these

14. If $\cos \theta = -\frac{12}{13}$ then $\tan \theta$ is

(a) $-\frac{12}{13}$ but not $\frac{12}{15}$ (b) $-\frac{12}{5}$ and $\frac{12}{5}$
(c) $\frac{12}{5}$ and $-\frac{12}{5}$ (d) None of these

15. The value of $\tan 380^0 \cot 20^0$ is

(a) 0 (b) 1 (c) $\tan^2 20^0$ (d) $\cot^2 20^0$

16. The value of the expression $\frac{\sin 70^0}{\sin 110^0}$ is

(a) 2 (b) 0 (c) 1 (d) None of these

17. $\frac{\cos 17^0 + \sin 17^0}{\cos 17^0 - \sin 17^0} =$

(a) $\tan 17^0$ (b) $\tan 62^0$ (c) $\tan 29^0$ (d) None of these

18. A tower is $200\sqrt{3}$ m high then the angle of elevation of its top from a point 200m away from its foot is:

(a) 30^0 (b) 60^0 (c) 45^0 (d) None of these

ANSWERS

1. $20\sqrt{3}$ 2. 19 m 3. $2\sqrt{3}$m 4. 14.1 m 5. 87 m

6. 51.24 m and 70 m 7. 10 metres 8. h = 17.32 m, breadth 9. (b) 10. (b)

11. (c) 12. (a) 13. (d) 14. (b) 15. (b) 16. (c)

17. (b) 18. (b)

UNIT 3
CO-ORDINATE GEOMETRY

Learning Objectives

- To understand and identify features of two dimensional figures; point, straight line and circle.
- To learn different forms of straight line and circle with different methods to solve them.
- Understand the basic concepts of two dimensional coordinate geometry with point, straight line and circle.

3.1 POINT

Cartesian Plane: Let XOX' and YOY' be two perpendicular lines. ‘O’ be their intersecting point called origin. XOX' is horizontal line called X-axis and YOY' is vertical line called Y-axis. The plane made by these axes is called Cartesian plane or coordinate plane.

The axes divide the plane into four parts called quadrant: 1st quadrant, 2nd quadrant, 3rd quadrant and 4th quadrant as shown in the figure. OX is known as positive direction of X-axis and OX' is known as negative direction of X-axis

Similarly, OY is known as positive direction of Y-axis and OY' is known as negative direction of Y-axis.

The axes XOX' and YOY' are together known as rectangular axes or coordinate axes.

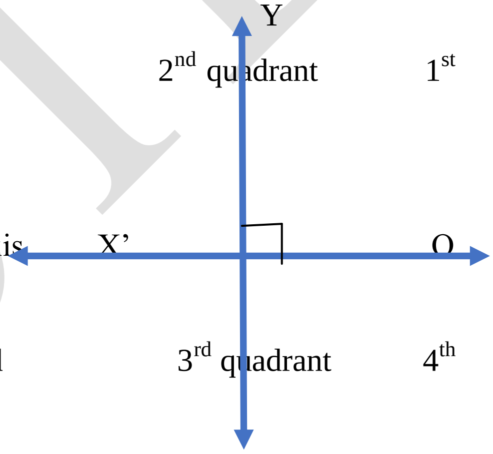
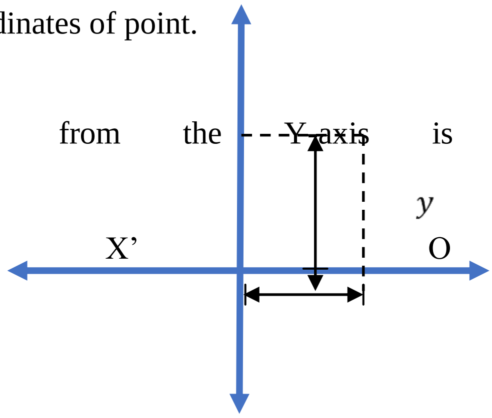


Fig. 3.1

Point: A point is a mark of location on a plane. It has no dimensions i.e. no length, no breadth and no height. For example, tip of pencil, toothpick etc. A point in a plane is represented as an ordered pair of real numbers called coordinates of point.

The perpendicular distance of a point from the Y-axis is called abscissa or x-coordinate and the perpendicular distance of a point from the X-axis is called ordinate or y-coordinate. If $P(x,y)$ be any point in the plane then x is the abscissa of the point P and y is the ordinate of the



point P.

Y'

Fig. 3.2

Note: **(i)** If distance along X-axis is measured to the right of Y-axis then it is positive and if it is measured to the left of Y-axis then it is negative.

(ii) If distance along Y-axis is measured to the above of X-axis then it is positive and if it is measured to the below of X-axis then it is negative.

(iii) The coordinates of origin 'O' are $(0,0)$.

(iv) A point on X-axis is represented as $(x,0)$ i.e. ordinate is zero.

(v) A point on Y-axis is represented as $(0,y)$ i.e. abscissa is zero.

(vi) In 1st quadrant $x>0$ and $y>0$

In 2nd quadrant $x<0$ and $y>0$

In 3rd quadrant $x<0$ and $y<0$

In 4th quadrant $x>0$ and $y<0$.

Distance between Two Points in a Plane: Let $A(x_1,y_1)$ and $B(x_2,y_2)$ be any two points in a plane then the distance between them is given by

$$AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Example1. Plot the following points and find the quadrant in which they lie:

(i) $A(2,2)$ **(ii)** $B(-3,-1)$ **(iii)** C$(-1,3)$ **(iv)** $D(3,-2)$

Sol.

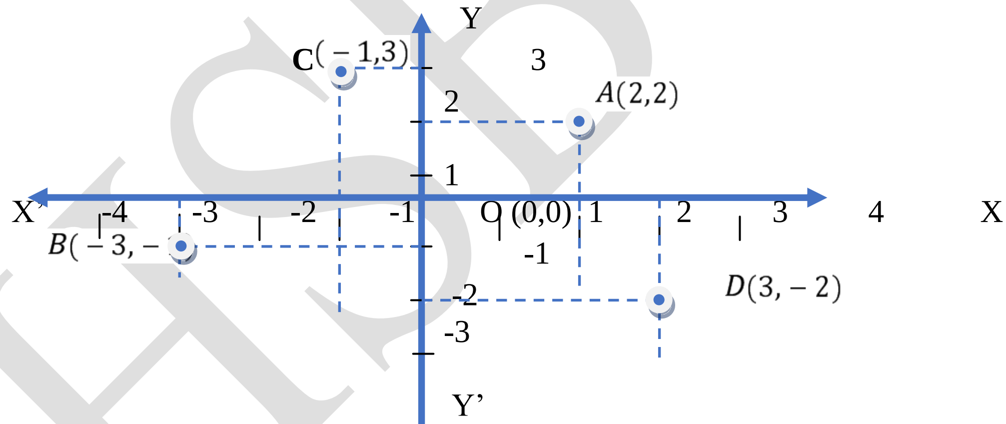


Fig. 3.3

By graph it is clear that

(i) Point $A(2,2)$ lies in the 1st quadrant.

(ii) Point $B(-3,-1)$ lies in the 3rd quadrant.

(iii) Point $C(-1,3)$ lies in the 2nd quadrant.

(iv) Point $D(3,-2)$ lies in the 4th quadrant.

Example 2. Without plotting, find the quadrant in which the following points lie:

(i) $A(2,-3)$ (ii) $B(-5,-6)$ (iii) C$(4,3)$ (iv) $D(-1,5)$

(v) $E(0,9)$ (vi) $F(-3,0)$ (vii) G$(0,-7)$ (viii) $H(1,0)$

Sol.

(i) The given point is $A(2,-3)$

Here X-coordinate = 2, which is positive and Y-coordinate = -3, which is negative.

Hence the point $A(2,-3)$ lies in 4th quadrant.

(ii) The given point is $B(-5,-6)$

Here X-coordinate = -5, which is negative and Y-coordinate = -6, which is also negative.

Hence the point $B(-5,-6)$ lies in 3rd quadrant.

(iii) The given point is C$(4,3)$

Here X-coordinate = 4 > 0 and Y-coordinate = 3 > 0.

Hence the point C$(4,3)$ lies in 1st quadrant.

(iv) The given point is $D(-1,5)$

Here X-coordinate = -1 < 0 and Y-coordinate = 5 > 0.

Hence the point $D(-1,5)$ lies in 2nd quadrant.

(v) The given point is $E(0,9)$

Here X-coordinate = 0 and Y-coordinate = 9 > 0.

Hence the point $E(0,9)$ lies on Y-axis above the origin.

(vi) The given point is $F(-3,0)$

Here X-coordinate = -3 < 0 and Y-coordinate = 0.

Hence the point $F(-3,0)$ lies on X-axis left to origin.

(vii) The given point is G$(0,-7)$

Here X-coordinate = 0 and Y-coordinate = -7 < 0.

Hence the point G$(0,-7)$ lies on Y-axis below the origin.

(viii) The given point is $H(1,0)$

Here X-coordinate = 1 > 0 and Y-coordinate = 0.

Hence the point $H(1,0)$ lies on X-axis right to origin.

Example3. Find the distance between the following pairs of points:

(i) $(0,5)$, $(3,6)$ (ii) $(-1,2)$, $(4,3)$ (iii) $(2,0)$, $(-3,-2)$

(iv) $(1,2)$, $(4,5)$ (v) $(-2,3)$, $(-5,7)$ (vi) $(-1,-3)$, $(-2,-4)$

(vii) $(a-b,c-d)$, $(-b+c,c+d)$ (viii) $(sin\theta,cos\theta)$, $(-sin\theta,cos\theta)$

Sol.

(i) Let **A** represents the point $(0,5)$ and **B** represents the point $(3,6)$.

So, the distance between **A** and **B** is:

$$AB=\sqrt{(3-0)^2+(6-5)^2}$$

$$=\sqrt{(3)^2+(1)^2}=\sqrt{9+1}=\sqrt{10}\ units$$

(ii) Let **A** represents the point $(-1,2)$ and **B** represents the point $(4,3)$.

So, the distance between **A** and **B** is:

$$AB=\sqrt{(4-(-1))^2+(3-2)^2}$$

$$=\sqrt{(5)^2+(1)^2}=\sqrt{25+1}=\sqrt{26}\ units$$

(iii) Let **A** represents the point $(2,0)$ and **B** represents the point $(-3,-2)$.

So, the distance between **A** and **B** is:

$$AB=\sqrt{(-3-2)^2+(-2-0)^2}$$

$$=\sqrt{(-5)^2+(-2)^2}=\sqrt{25+4}=\sqrt{29}\ units$$

(iv) Let **A** represents the point $(1,2)$ and **B** represents the point $(4,5)$.

So, the distance between **A** and **B** is:

$$AB=\sqrt{(4-1)^2+(5-2)^2}$$

$$=\sqrt{(3)^2+(3)^2}=\sqrt{9+9}$$
$$=\sqrt{18}=\sqrt{9\times 2}=3\sqrt{2}\ units$$

(v) Let **A** represents the point $(-2,3)$ and **B** represents the point $(-5,7)$.
So, the distance between **A** and **B** is:
$$AB=\sqrt{(-5-(-2))^2+(7-3)^2}$$
$$=\sqrt{(-5+2)^2+(4)^2}=\sqrt{(-3)^2+(4)^2}$$
$$=\sqrt{9+16}=\sqrt{25}=5\,units$$

(vi) Let **A** represents the point $(-1,-3)$ and **B** represents the point $(-2,-4)$.
So, the distance between **A** and **B** is:
$$AB=\sqrt{(-2-(-1))^2+(-4-(-3))^2}$$
$$=\sqrt{(-2+1)^2+(-4+3)^2}=\sqrt{(-1)^2+(-1)^2}$$
$$=\sqrt{1+1}=\sqrt{2}\ units$$

(vii) Let **A** represents the point $(a-b,c-d)$ and **B** represents the point $(-b+c,c+d)$.
So, the distance between **A** and **B** is:
$$AB=\sqrt{(-b+c-(a-b))^2+(c+d-(c-d))^2}$$
$$=\sqrt{(-b+c-a+b)^2+(c+d-c+d)^2}$$
$$=\sqrt{(c-a)^2+(d+d)^2}=\sqrt{c^2+a^2-2ac+(2d)^2}$$
$$=\sqrt{c^2+a^2+4d^2-2ac}\ \ units$$

(viii) Let **A** represents the point $(sin\theta,cos\theta)$ and **B** represents the point $(-sin\theta,cos\theta)$.
So, the distance between **A** and **B** is:
$$AB=\sqrt{(-\sin\theta-\sin\theta)^2+(\cos\theta-\cos\theta)^2}$$
$$=\sqrt{(-2\sin\theta)^2+(0)^2}=\sqrt{4\sin^2\theta}$$
$$=2\sin\theta\,units$$

Example 4. Using distance formula, prove that the triangle formed by the points $A(4,0)$, $B(-1,-1)$ and $C(3,5)$ is an isosceles triangle.

Sol. Given that vertices of the triangle are $A(4,0)$, $B(-1,-1)$ and $C(3,5)$.
To find the length of edges of the triangle, we will use the distance formula:
Distance between **A** and **B** is
$$AB=\sqrt{(4-(-1))^2+(0-(-1))^2}$$
$$=\sqrt{(5)^2+(1)^2}=\sqrt{25+1}=\sqrt{26}\,units$$
Distance between **B** and **C** is
$$BC=\sqrt{(-1-3)^2+(-1-5)^2}$$
$$=\sqrt{(-4)^2+(-6)^2}=\sqrt{16+36}=\sqrt{52}\,units$$
Distance between **A** and **C** is
$$AC=\sqrt{(4-3)^2+(0-5)^2}$$
$$=\sqrt{(1)^2+(-5)^2}=\sqrt{1+25}=\sqrt{26}\,units$$
We can see that $AB=AC\neq BC$

Hence the triangle formed by the points $A(4,0)$, $B(-1,-1)$ and $C(3,5)$ is an isosceles triangle.

Example 5. Using distance formula, prove that the triangle formed by the points $A(0,0)$, $B(0,2)$ and $C(\sqrt{3},1)$ is an equilateral triangle.

Sol. Given that vertices of the triangle are $A(0,0)$, $B(0,2)$ and $C(\sqrt{3},1)$.

To find the length of edges of the triangle, we will use the distance formula:

Distance between **A** and **B** is

$$AB=\sqrt{(0-0)^2+(0-2)^2}$$

$$=\sqrt{(0)^2+(-2)^2}=\sqrt{0+4}=\sqrt{4}=2\,units$$

Distance between **B** and **C** is

$$BC=\sqrt{\left(0-\sqrt{3}\right)^2+(2-1)^2}$$

$$=\sqrt{\left(-\sqrt{3}\right)^2+(1)^2}=\sqrt{3+1}=\sqrt{4}=2\,units$$

Distance between **A** and **C** is

$$AC=\sqrt{\left(0-\sqrt{3}\right)^2+(0-1)^2}$$

$$=\sqrt{\left(-\sqrt{3}\right)^2+(-1)^2}=\sqrt{3+1}=\sqrt{4}=2\,units$$

We can see that $AB=BC=AC$

Hence the triangle formed by the points $A(0,0)$, $B(0,2)$ and $C(\sqrt{3},1)$ is an equilateral triangle.

Example 6. Find the mid points between the following pairs of points:

(i) $(2,3)$, $(8,5)$ (ii) $(6,3)$, $(6,-9)$ (iii) $(-2,-4)$, $(3,-6)$

(iv) $(0,8)$, $(6,0)$ (v) $(0,0)$, $(-12,10)$ (vi) (a,b), (c,d)

(vii) $(a+b,c-d)$, $(-b+3a,c+d)$

Sol.

(i) The given points are $(2,3)$ and $(8,5)$.

So, the mid-point between these points is given by:

$$\left(\frac{2+8}{2},\frac{3+5}{2}\right)=\left(\frac{10}{2},\frac{8}{2}\right)=(5,4)$$

(ii) The given points are $(6,3)$ and $(6,-9)$.

So, the mid-point between these points is given by:

$$\left(\frac{6+6}{2},\frac{3+(-9)}{2}\right)=\left(\frac{12}{2},\frac{3-9}{2}\right)=\left(\frac{12}{2},\frac{-6}{2}\right)=(6,-3)$$

(iii) The given points are $(-2,-4)$ and $(3,-6)$.

So, the mid-point between these points is given by:

$$\left(\frac{-2+3}{2},\frac{-4+(-6)}{2}\right)=\left(\frac{1}{2},\frac{-4-6}{2}\right)=\left(\frac{1}{2},\frac{-10}{2}\right)=\left(\frac{1}{2},-5\right)$$

(iv) The given points are $(0,8)$ and $(6,0)$.

So, the mid-point between these points is given by:

$$\left(\frac{0+6}{2},\frac{8+0}{2}\right)=\left(\frac{6}{2},\frac{8}{2}\right)=(3,4)$$

(v) The given points are $(0,0)$ and $(-12,10)$.

So, the mid-point between these points is given by:

$$\left(\frac{0+(-12)}{2},\frac{0+10}{2}\right)=\left(\frac{-12}{2},\frac{10}{2}\right)=(-6,5)$$

(vi) The given points are (a,b) and (c,d).

So, the mid-point between these points is given by:

$$\left(\frac{a+c}{2},\frac{b+d}{2}\right)$$

(vii) The given points are $(a+b,c-d)$ and $(-b+3a,c+d)$.

So, the mid-point between these points is given by:

$$\left(\frac{a+b-b+3a}{2},\frac{c-d+c+d}{2}\right)=\left(\frac{4a}{2},\frac{2c}{2}\right)=(2a,c)$$

Example 7. If the mid-point between two points is $(3,5)$ and one point between them is $(-1,2)$, find the other point.

Sol. Let the required point is (a,b).

According to given statement $(3,5)$ is the mid-point of $(-1,2)$ and (a,b).

$$\Rightarrow \quad (3,5)=\left(\frac{-1+a}{2},\frac{2+b}{2}\right)$$

$$\Rightarrow \quad \frac{-1+a}{2}=3 \quad \& \quad \frac{2+b}{2}=5$$

$$\Rightarrow \quad -1+a=6 \quad \& \quad 2+b=10$$

$$\Rightarrow \quad a=7 \quad \& \quad b=8$$

Hence the required point is $(7,8)$.

Example8. If the mid-point between two points is $(-7,6)$ and one point between them is $(3,-9)$, find the other point.

Sol. Let the required point is (a,b).

According to given statement $(-7,6)$ is the mid-point of $(3,-9)$ and (a,b).

$$\Rightarrow \quad (-7,6)=\left(\frac{3+a}{2},\frac{-9+b}{2}\right)$$

$$\Rightarrow \quad \frac{3+a}{2}=-7 \quad \& \quad \frac{-9+b}{2}=6$$

$$\Rightarrow \quad 3+a=-14 \quad \& \quad -9+b=12$$

$$\Rightarrow \quad a=-17 \quad \& \quad b=21$$

Hence the required point is $(-17,21)$.

Centroid of a Triangle: The centroid of a triangle is the intersection point of the three medians of the triangle. In other words, the **average** of the three vertices of the triangle is called the centroid of the triangle.

i.e. If $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are three vertices of a triangle then the centroid of the triangle is given by:

$$\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$$

In this fig.3.4, the point G is the centroid of the triangle.

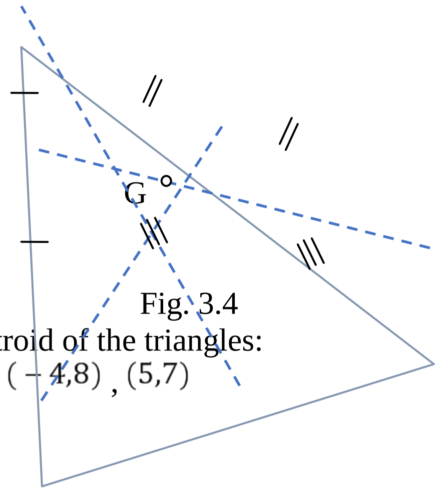


Fig. 3.4

Example9. Vertices of the triangles are given below, find the centroid of the triangles:

(i) $(5,2)$, $(5,4)$, $(8,6)$ (ii) $(4,-3)$, $(-4,8)$, $(5,7)$

(iii) $(2,-4)$, $(0,-10)$, $(4,5)$ (iv) $(9,-9)$, $(5,8)$, $(-7,-2)$

Sol.

(i) The given vertices of the triangle are $(5,2)$, $(5,4)$ and $(8,6)$.
So, the centroid of the triangle is

$$\left(\frac{5+5+8}{3},\frac{2+4+6}{3}\right)=\left(\frac{18}{3},\frac{12}{3}\right)=(6,4)$$

(ii) The given vertices of the triangle are $(4,-3)$, $(-4,8)$ and $(5,7)$.
So, the centroid of the triangle is

$$\left(\frac{4-4+5}{3},\frac{-3+8+7}{3}\right)=\left(\frac{5}{3},\frac{12}{3}\right)=\left(\frac{5}{3},4\right)$$

(iii) The given vertices of the triangle are $(2,-4)$, $(0,-10)$ and $(4,5)$.
So, the centroid of the triangle is

$$\left(\frac{2+0+4}{3},\frac{-4-10+5}{3}\right)=\left(\frac{6}{3},\frac{-9}{3}\right)=(2,-3)$$

(iv) The given vertices of the triangle are $(9,-9)$, $(5,8)$ and $(-7,-2)$.
So, the centroid of the triangle is

$$\left(\frac{9+5-7}{3},\frac{-9+8-2}{3}\right)=\left(\frac{7}{3},\frac{-3}{3}\right)=\left(\frac{7}{3},-1\right)$$

Example 10. If centroid of the triangle is $(10,18)$ and two vertices of the triangle are $(1,-5)$ and $(3,7)$, find the third vertex of the triangle.

Sol. Let the required vertex of the triangle is (a,b).
So, according to given statement and definition of centroid, we get

$$\Rightarrow \quad (10,18)=\left(\frac{1+3+a}{3},\frac{-5+7+b}{3}\right)$$

$$\Rightarrow \quad \frac{1+3+a}{3}=10 \quad \& \quad \frac{-5+7+b}{3}=18$$

$$\Rightarrow \quad 1+3+a=30 \quad \& \quad -5+7+b=54$$

$$\Rightarrow \quad a=26 \quad \& \quad b=52$$

Hence the required vertex of triangle is $(26,52)$.

Example 11. If centroid of the triangle is $(-5,-7)$ and two vertices of the triangle are $(0,6)$ and $(-3,2)$, find the third vertex of the triangle.

Sol. Let the required vertex of the triangle is (a,b).
So, according to given statement and definition of centroid, we get

$$\Rightarrow \quad (-5,-7)=\left(\frac{0-3+a}{3},\frac{6+2+b}{3}\right)$$

$$\Rightarrow \quad \frac{0-3+a}{3}=-5 \quad \& \quad \frac{6+2+b}{3}=-7$$

$$\Rightarrow \quad 0-3+a=-15 \quad \& \quad 6+2+b=-21$$

$$\Rightarrow \quad a=-12 \quad \& \quad b=-29$$

Hence the required vertex of triangle is $(-12,-29)$.

Example 12. If centroid of a triangle formed by the points $(1,a)$, $(9,b)$ and $(c^2,-5)$ lies on the X-axis, prove that $a+b=5$.

Sol. Given that vertices of the triangle are $(1,a)$, $(9,b)$ and $(c^2,-5)$.
Centroid of the triangle is given by

$$\left(\frac{1+9+c^2}{3},\frac{a+b-5}{3}\right)$$

By given statement centroid lies on the X-axis. Therefore, Y-coordinate of centroid is zero

$$\Rightarrow \qquad \frac{a+b-5}{3}=0$$

$$\Rightarrow \qquad a+b-5=0$$

$$\Rightarrow \qquad a+b=5$$

Example 13. If centroid of a triangle formed by the points $(-a,a)$, (c^2,b) and $(d,5)$ lies on the Y-axis, prove that $c^2=a-d$.

Sol. Given that vertices of the triangle are $(-a,a)$, (c^2,b) and $(d,5)$.

Centroid of the triangle is given by

$$\left(\frac{-a+c^2+d}{3},\frac{a+b+5}{3}\right)$$

By given statement centroid lies on the Y-axis. Therefore, X-coordinate of centroid is zero

$$\Rightarrow \qquad \frac{-a+c^2+d}{3}=0$$

$$\Rightarrow \qquad -a+c^2+d=0$$

$$\Rightarrow \qquad c^2=a-d$$

Hence proved.

<u>Area of a Triangle with given vertices</u>:

If (x_1,y_1), (x_2,y_2) and (x_3,y_3) are vertices of a triangle then area of triangle is given by

$$\Delta=\pm\frac{1}{2}\left[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_1y_3)\right]$$

To remember this we can take help of figure given below:

$$\Delta=\pm\frac{1}{2}\left|\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{array}\right|$$

<u>Alternate Method</u>: We can also find the area of triangle by the use of determinant if all the three vertices are given. If (x_1,y_1), (x_2,y_2) and (x_3,y_3) are vertices of a triangle then area of triangle is given by

$$\Delta=\pm\frac{1}{2}\begin{vmatrix} x_1-x_2 & x_1-x_3 \\ y_1-y_2 & y_1-y_3 \end{vmatrix}$$

Note: **(i)** Area is always non-negative. So take the suitable sign that gives the non-negative value.

(ii) If $\Delta=0$ then the three points don't form triangle and these are **collinear** points.

Example 14. Vertices of the triangles are given below, find the area of the triangles:

(i) $(3,2)$, $(5,4)$, $(7,2)$ (ii) $(1,3)$, $(-4,5)$, $(3,-4)$

(iii) $(-2,1)$, $(2,-3)$, $(5,5)$ (iv) $(0,2)$, $(3,6)$, $(7,-5)$

Sol.

(i) The given vertices of the triangle are $(3,2)$, $(5,4)$ and $(7,2)$.

Comparing these points with (x_1,y_1), (x_2,y_2) and (x_3,y_3) respectively, we get
$x_1 = 3, y_1 = 2$, $x_2 = 5, y_2 = 4$, $x_3 = 7, y_3 = 2$

So, area of the triangle is given by

$$\Delta = \pm\frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\right]$$

i.e. $$\Delta = \pm\frac{1}{2}\left[(3\times 4 - 5\times 2) + (5\times 2 - 7\times 4) + (7\times 2 - 3\times 2)\right]$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[(12-10) + (10-28) + (14-6)\right]$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[2-18+8\right] = \pm\frac{1}{2}\left[-8\right] = 4\, sq.units$$

Alternate Method

The given vertices of the triangle are $(3,2)$, $(5,4)$ and $(7, 2)$.

Comparing these points with (x_1,y_1), (x_2,y_2) and (x_3,y_3) respectively, we get
$x_1 = 3, y_1 = 2$, $x_2 = 5, y_2 = 4$, $x_3 = 7, y_3 = 2$

So, area of the triangle is given by

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} x_1 - x_2 & x_1 - x_3 \\ y_1 - y_2 & y_1 - y_3 \end{vmatrix}$$

i.e. $$\Delta = \pm\frac{1}{2}\begin{vmatrix} 3-5 & 3-7 \\ 2-4 & 2-2 \end{vmatrix}$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\begin{vmatrix} -2 & -4 \\ -2 & 0 \end{vmatrix}$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[(-2)(0) - (-2)(-4)\right]$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[0-8\right] = 4\, sq.units$$

(ii) The given vertices of the triangle are $(1,3)$, $(-4,5)$ and $(3,-4)$.

Comparing these points with (x_1,y_1), (x_2,y_2) and (x_3,y_3) respectively, we get
$x_1 = 1, y_1 = 3$, $x_2 = -4, y_2 = 5$, $x_3 = 3, y_3 = -4$

So, area of the triangle is given by

$$\Delta = \pm\frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\right]$$

i.e. $$\Delta = \pm\frac{1}{2}\left[\{1\times 5 - (-4)\times 3\} + \{-4\times(-4) - 3\times 5\} + \{3\times 3 - 1\times(-4)\}\right]$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[(5+12) + (16-15) + (9+4)\right]$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[17+1+13\right] = \pm\frac{1}{2}\left[31\right] = 15.5\, sq.units$$

(iii) The given vertices of the triangle are $(-2,1)$, $(2,-3)$ and $(5,5)$.

Comparing these points with (x_1,y_1), (x_2,y_2) and (x_3,y_3) respectively, we get
$x_1 = -2, y_1 = 1$, $x_2 = 2, y_2 = -3$, $x_3 = 5, y_3 = 5$

So, area of the triangle is given by

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

i.e.

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} -2 & 1 \\ 2 & -3 \\ 5 & 5 \\ -2 & 1 \end{vmatrix}$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[(6-2)+(10-(-15))+(5+10)\right]$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}[4+25+15] = \pm\frac{1}{2}[44] = 22\,sq.units$$

(iv) The given vertices of the triangle are $(0,2)$, $(3,6)$ and $(7,-5)$.

Comparing these points with (x_1,y_1), (x_2,y_2) and (x_3,y_3) respectively, we get $x_1 = 0, y_1 = 2$, $x_2 = 3, y_2 = 6$, $x_3 = 7, y_3 = -5$

So, area of the triangle is given by

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

i.e.

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} 0 & 2 \\ 3 & 6 \\ 7 & -5 \\ 0 & 2 \end{vmatrix}$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[(0-6)+(-15-42)+(14-0)\right]$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}[-6-57+14] = \pm\frac{1}{2}[-49] = 24.5\,sq.units$$

Example 15. If the area of the triangle with vertices $(1,1)$, $(x,0)$ **,** $(0,2)$ is 4 sq. units, find the value of x.

Sol. The given vertices of the triangle are $(1,1)$, $(x,0)$ and $(0,2)$.

Comparing these points with (x_1,y_1), (x_2,y_2) and (x_3,y_3) respectively, we get $x_1 = 1, y_1 = 1$, $x_2 = x, y_2 = 0$, $x_3 = 0, y_3 = 2$

Also it is given that the area of the triangle is 4 s**q.** units

i.e. $\Delta = 4\,sq.units$ (1)

Now, area of the triangle is given by

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

i.e.

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} 1 & 1 \\ x & 0 \\ 0 & 2 \\ 1 & 1 \end{vmatrix}$$

$\Rightarrow \qquad \Delta = \pm\frac{1}{2}\left[(0-x)+(2x-0)+(0-2)\right]$

$\Rightarrow \qquad \Delta = \pm\frac{1}{2}\left[-x+2x-2\right] = \pm\frac{1}{2}\left[x-2\right] sq.units$ (2)

Comparing (1) and (2), we get

$$\pm\frac{1}{2}\left[x-2\right]=4$$

$\Rightarrow \qquad \pm\left[x-2\right]=8$

$\Rightarrow \qquad$ *either* $x-2=8$ *or* $x-2=-8$

$\Rightarrow \qquad$ *either* $x=10$ *or* $x=-6$

which is the required sol.

Example 16. Prove that the triplet of points $(4,7)$, $(0,1)$ **,** $(2, 4)$ is collinear.

Sol. The points are $(4,7)$, $(0,1)$ and $(2, 4)$.

Comparing these points with (x_1,y_1), (x_2,y_2) and (x_3,y_3) respectively, we get

$x_1 = 4, y_1 = 7$, $x_2 = 0, y_2 = 1$, $x_3 = 2, y_3 = 4$

So by formula of area of the triangle, we get

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

i.e.

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} 4 & 7 \\ 0 & 1 \\ 2 & 4 \\ 4 & 7 \end{vmatrix}$$

$\Rightarrow \qquad \Delta = \pm\frac{1}{2}\left[(4-0)+(0-2)+(14-16)\right]$

$\Rightarrow \qquad \Delta = \pm\frac{1}{2}\left[4-2-2\right]=0$

which shows that the given points are collinear.

Example 17. Find the value of x, in order that the points $(5,-1)$, $(-4,2)$ **and** $(x, 6)$ are collinear.

Sol. The given points are $(5,-1)$, $(-4,2)$ and $(x, 6)$.

Comparing these points with (x_1,y_1), (x_2,y_2) and (x_3,y_3) respectively, we get

$x_1 = 5, y_1 = -1$, $x_2 = -4, y_2 = 2$, $x_3 = x, y_3 = 6$

Also it is given that the points $(5,-1)$, $(-4,2)$ and $(x, 6)$ are collinear

i.e. $\Delta=0$ (1)

Now, area of the triangle is given by

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

i.e.

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} 5 & -1 \\ -4 & 2 \\ x & 6 \\ 5 & -1 \end{vmatrix}$$

$\Rightarrow$ $\Delta = \pm\frac{1}{2}\left[(10-4)+(-24-2x)+(-x-30)\right]$

$\Rightarrow$ $\Delta = \pm\frac{1}{2}\left[6-24-2x-x-30\right] = \pm\frac{1}{2}\left[-3x-48\right] sq.units$ (2)

Comparing (1) and (2), we get

$$\pm\frac{1}{2}\left[-3x-48\right]=0$$

$\Rightarrow$ $-3x-48=0$

$\Rightarrow$ $-3x=48$

$\Rightarrow$ $x=-\frac{48}{3}$

$\Rightarrow$ $x=-16$

which is the required solution.

EXERCISE – I

1. The point (-3, -4) lies in quadrant:

   (a) First (b) Second (c) Third (d) Fourth

2. Three points are collinear and Δ is the area of triangle formed with these three points, then

   (a) $\Delta = 0$ (b) $\Delta > 0$ (c) $\Delta < 0$ (d) 1

3. Find the distance between the following pairs of points:

   (i) (-1, 2), (4, 3) (ii) (a - b, c - d), (-b + c, c + d)

4. Find the mid points between the following pairs of points:

   (i) (0, 8), (6, 0) (ii) (a + b, c - d), (-b + 3a, c + d)

5. The midpoint between two points is (3, 5) and one point between them is (-1, 2). Find the other point.

6. Find the centroid of the triangle whose vertices are:

   (i) (4, -3), (-4, 8), (5, 7) (ii) (9, -9), (5, 8), (-7, -2)

7. Find the area of the triangles whose vertices are:

(i) (1, 3), (-4, 5), (3, -4) (ii) (0, 2), (3, 6), (7, -5)

8. Prove that the triplet of points (4, 7), (0, 1), (2, 4) is collinear.

ANSWERS

1. (d) 2. (a) 3. (i) $[\sqrt{26}]$ (ii) $[\sqrt{c^2 + a^2 + 4d^2 - 2ac}\,]$ 4. (i) (3,4) (ii) (2a, c)

5. (7, 8) 6. (i) (5/3, 4) (ii) (7/3, -1) 7. (i) 15.5 (ii) 24.5

3.2 STRAIGHT LINE

Definition: A path traced by a point travelling in a constant direction is called a straight line.
OR
The shortest path between two points is called a straight line.

General Equation of Straight Line: A straight line in XY plane has general form

$$ax + by + c = 0$$

where a is the coefficient of x, b is the coefficient of y and c is the constant term so that at least one of a,b is non-zero.

Note: **(i)** Any point (x_1, y_1) lies on the line $ax + by + c = 0$ if it satisfies the equations of the line *i.e.* if we substitute the values x_1 at the place of x and y_1 at the place of y in the equation of line, the result $ax_1 + by_1 + c$ becomes zero.

(ii) X-axis is usually represented horizontally and its equation is $y = 0$.

(iii) Y-axis is usually represented vertically and its equation is $x = 0$.

(iv) $x = k$ represents the line parallel to Y-axis, where k is some constant .

(v) $y = k$ represents the line parallel to X-axis, where k is some constant .

Slope of a Straight Line: Slope of straight line measures with tangent of the angle of straight line to the horizon.
It is usually represented by m.
X

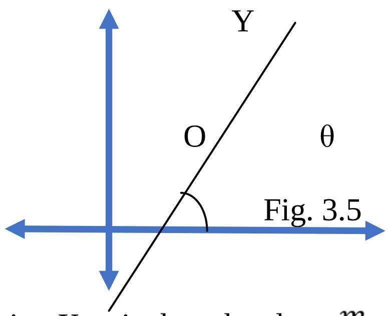


Fig. 3.5

To find slope of a straight Line:

(i) If a non-vertical line making an angle θ with positive X-axis then the slope m of the line is given by $m = tan\theta$.

(ii) If a non-vertical line passes through two points (x_1, y_1) and (x_2, y_2) then the slope m of the line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

(iii) If equation of a straight line is $ax + by + c = 0$, then its slope m is given by $m = -\frac{a}{b}$.

Note: **(i)** Slop of a horizontal line is always zero **i.e.,** slope of a line parallel to X-axis is zero as $m = tan 0° = 0$.

(ii) Slop of a vertical line is always infinity **i.e.,** slope of a line perpendicular to X-axis is infinity as $= tan 90° = \infty$.

(iii) Let L_1 and L_2 represents two straight lines. Let m_1 and m_2 be slopes of L_1 and L_2 respectively. We say that L_1 and L_2 are parallel lines iff $m_1 = m_2$ **i.e.** slopes are equal. We say that L_1 and L_2 are perpendicular iff $m_1 . m_2 = -1$ **i.e.,** product of slopes is equal to -1.

Example 18. Find the slope of the straight lines which make following angles:

(i) 45° (ii) 120° (iii) 30° (iv) 150° (v) 210°

with the positive direction of X-axis.

Sol.

(i) Let m be the slope of the straight line and θ be the angle which the straight line makes with the positive direction of X-axis.

Therefore $\theta = 45°$ and $m = tan\theta$

$\Rightarrow \quad m = \tan 45^\circ$

$\Rightarrow \quad m = 1$

which is the required slope.

(ii) Let m be the slope of the straight line and θ be the angle which the straight line makes with the positive direction of X-axis.

Therefore $\theta = 120°$ and $m = tan\theta$

$\Rightarrow \quad m = \tan 120^\circ$

$\Rightarrow \quad m = \tan\left(180^\circ - 60^\circ\right)$

$\Rightarrow \quad m = -\tan\left(60^\circ\right)$

$\Rightarrow \quad m = -\sqrt{3}$

which is the required slope.

(iii) Let m be the slope of the straight line and θ be the angle which the straight line makes with the positive direction of X-axis.

Therefore $\theta = 30°$ and $m = tan\theta$

$\Rightarrow \quad m = \tan 30^\circ$

$\Rightarrow \quad m = \frac{1}{\sqrt{3}}$

which is the required slope.

(iv) Let m be the slope of the straight line and θ be the angle which the straight line makes with the positive direction of X-axis.

Therefore $\theta = 150°$ and $m = tan\theta$

$\Rightarrow \quad m = \tan 150^\circ$

$\Rightarrow \quad m = \tan\left(180^\circ - 30^\circ\right)$

$\Rightarrow \quad m = -\tan\left(30^\circ\right)$

$\Rightarrow \quad m = -\frac{1}{\sqrt{3}}$

which is the required slope.

(v) Let m be the slope of the straight line and θ be the angle which the straight line makes with the positive direction of X-axis.

Therefore $\theta = 210°$ and $m = tan\theta$

$\Rightarrow \quad m = \tan 210^\circ$

$\Rightarrow \quad m = \tan(180^\circ + 30^\circ)$

$\Rightarrow \quad m = \tan(30^\circ)$

$\Rightarrow \quad m = \dfrac{1}{\sqrt{3}}$

which is the required slope.

Example 19. Find the slope of the straight lines which pass through the following pairs of points:

(i) $(2,5)$, $(6,17)$ (ii) $(-8,7)$, $(3,-5)$ (iii) $(0,-6)$, $(7,9)$
(iv) $(-11,-5)$, $(-3,-10)$ (v) $(0,0)$, $(10,-12)$.

Sol.

(i) Given that the straight line passes through the points $(2,5)$ and $(6,17)$.

Comparing these points with (x_1,y_1) and (x_2,y_2) respectively, we get
$x_1 = 2$, $y_1 = 5$, $x_2 = 6$ and $y_2 = 17$

Let m be the slope of the straight line.

Therefore $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$\Rightarrow \quad m = \dfrac{17-5}{6-2} = \dfrac{12}{4}$

$\Rightarrow \quad m = 3$

which is the required slope.

(ii) Given that the straight line passes through the points $(-8,7)$ and $(3,-5)$.

Comparing these points with (x_1,y_1) and (x_2,y_2) respectively, we get
$x_1 = -8$, $y_1 = 7$, $x_2 = 3$ and $y_2 = -5$

Let m be the slope of the straight line.

Therefore $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$\Rightarrow \quad m = \dfrac{-5-7}{3-(-8)} = \dfrac{-12}{3+8}$

$\Rightarrow \quad m = -\dfrac{12}{11}$

which is the required slope.

(iii) Given that the straight line passes through the points $(0,-6)$ and $(7,9)$.

Comparing these points with (x_1,y_1) and (x_2,y_2) respectively, we get
$x_1 = 0$, $y_1 = -6$, $x_2 = 7$ and $y_2 = 9$

Let m be the slope of the straight line.

Therefore $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$\Rightarrow \quad m = \dfrac{9-(-6)}{7-0} = \dfrac{9+6}{7}$

$\Rightarrow \quad m = \dfrac{15}{7}$

which is the required slope.

(iv) Given that the straight line passes through the points $(-11,-5)$ and $(-3,-10)$.

Comparing these points with (x_1,y_1) and (x_2,y_2) respectively, we get

$x_1=-11$, $y_1=-5$, $x_2=-3$ and $y_2=-10$

Let m be the slope of the straight line.

Therefore $m=\dfrac{y_2-y_1}{x_2-x_1}$

$$\Rightarrow \quad m=\frac{-10-(-5)}{-3-(-11)}=\frac{-10+5}{-3+11}$$

$$\Rightarrow \quad m=-\frac{5}{8}$$

which is the required slope.

(v) Given that the straight line passes through the points $(0,0)$ and $(10,-12)$.

Comparing these points with (x_1,y_1) and (x_2,y_2) respectively, we get

$x_1=0$, $y_1=0$, $x_2=10$ and $y_2=-12$

Let m be the slope of the straight line.

Therefore $m=\dfrac{y_2-y_1}{x_2-x_1}$

$$\Rightarrow \quad m=\frac{-12-0}{10-0}=\frac{-12}{10}$$

$$\Rightarrow \quad m=-\frac{6}{5}$$

which is the required slope.

Example 20. Find the slopes of the following straight lines:

(i) $2x+4y+5=0$ (ii) $x-3y+9=0$

(iii) $5y-10x+1=0$ (iv) $-2x-6y=0$

(v) $x=5$ (vi) $y=-6$

Sol.

(i) Given that equation of the straight line is $2x+4y+5=0$.

Comparing this equation with $ax+by+c=0$, we get

$a=2$, $b=4$ and $c=5$

Let m be the slope of given straight line.

Therefore, $m=-\dfrac{a}{b}$

$$\Rightarrow \quad m=-\frac{2}{4}$$

$$\Rightarrow \quad m=-\frac{1}{2}$$

which is the required slope.

(ii) Given that equation of the straight line is $-3y+9=0$.

Comparing this equation with $ax+by+c=0$, we get

$a=1$, $b=-3$ and $c=9$

Let m be the slope of given straight line.

Therefore, $m=-\left(\frac{1}{-3}\right)$

$\Rightarrow \qquad m=\frac{1}{3}$

which is the required slope.

(iii) Given that equation of the straight line is $5y-10x+1=0$.

Comparing this equation with $ax+by+c=0$, we get

$a=-10$, $b=5$ and $c=1$

Let m be the slope of given straight line.

Therefore, $m=-\left(\frac{-10}{5}\right)$

$\Rightarrow \qquad m=2$

which is the required slope.

(iv) Given that equation of the straight line is $-2x-6y=0$.

Comparing this equation with $ax+by+c=0$, we get

$a=-2$, $b=-6$ and $c=0$

Let m be the slope of given straight line.

Therefore, $m=-\frac{a}{b}$

$\Rightarrow \qquad m=-\left(\frac{-2}{-6}\right)$

$\Rightarrow \qquad m=-\frac{1}{3}$

which is the required slope.

(v) Given that equation of the straight line is $=5$.

This equation is parallel to Y-axis.

Hence the slope of the line is infinity.

(vi) Given that equation of the straight line is $=-6$.

This equation is parallel to X-axis.

Hence the slope of the line is zero.

Example 21. Find the equation of straight line which is parallel to X-axis passes through $(1,5)$.

Sol. Equation of straight line parallel to X-axis is given by

$$y=k$$

Given that the straight line passes through the point $(1,5)$.

Put $x=1$ and $y=5$ in (1), we get

$$5=k$$

So, $y=5$ be the required equation of straight line.

Example 22. Find the equation of straight line which is parallel to Y-axis passes through $(-3,-7)$.

Sol. Equation of straight line parallel to Y-axis is given by

$$x=k$$

Given that the straight line passes through the point $(-3,-7)$.

Put $x=-3$ and $y=-7$ in (1), we get

$$-3=k$$

So, $x=-3$ be the required equation of straight line.

Equation of Straight Line Passing Through Origin:

If a non-vertical line passes through origin and m be its slope. $P(x,y)$ be any point on the line (see Fig. 3.6), then equation of straight line is $y = mx$

X

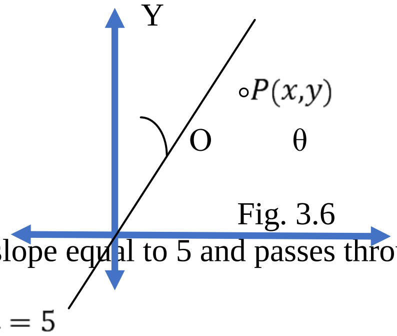


Fig. 3.6

Example 23. Find the equation of straight line having slope equal to 5 and passes through origin.

Sol. Let m be the slope of required line. Therefore $m = 5$.

Also it is given that the required line passes through the origin.We know that equation of straight line passes through origin is $y=mx$, where m be the slope of the line.

So, $y=5x$ be the required equation of straight line.

Example 24 Find the equation of straight line having slope equal to -10 and passes through origin.

Sol. Let m be the slope of required line. Therefore $m=-10$. Also it is given that the required line passes through the origin.

We know that equation of straight line passes through origin is $y=mx$, where m be the slope of the line. So, $y=-10x$ be the required equation of straight line.

Example 25. Find the equation of straight line which passes through origin and makes an angle $60°$ with the positive direction of X-axis.

Sol. Let m be the slope of required line.

Therefore $m = \tan 60^\circ$

$\Rightarrow \quad m = \sqrt{3}$

Also it is given that the required line passes through the origin. We know that equation of straight line passes through origin is $y=mx$, where m be the slope of the line.

So, $y=\sqrt{3}\,x$ be the required equation of straight line.

Example 26. Find the equation of straight line which passes through origin and makes an angle $135°$ with the positive direction of X-axis.

Sol. Let m be the slope of required line.

Therefore $m = \tan 135^\circ$

$\Rightarrow \quad m = \tan\left(180^\circ - 45^\circ\right)$

$\Rightarrow \quad m = -\tan 45^\circ$

$\Rightarrow \quad m = -1$

Also it is given that the required line passes through the origin.We know that equation of straight line passes through origin is $y=mx$, where m be the slope of the line.

So, $y=-x$ be the required equation of straight line.

Equation of Straight Line in Point-Slope form:

Let a non-vertical line passes through a point (x_1,y_1) and m be its slope. $P(x,y)$ be any point on the line (see fig. 3.7), then equationof straight line is $y - y_1 = m(x - x_1)$

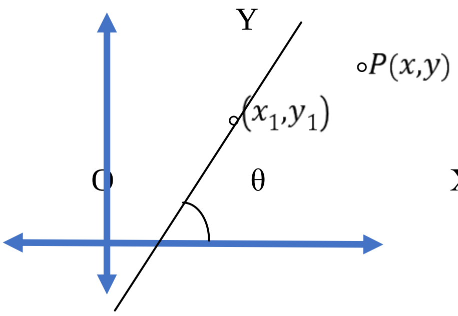


Fig. 3.7

Example 27. Find the equation of straight line having slope equal to 9 and passes through the point $(1,5)$.

Sol. Let m be the slope of required line. Therefore $m = 9$.

Also it is given that the required line passes through the point $(1,5)$. We know that equation of straight line in point slope form is $y - y_1 = m(x - x_1)$.

$\Rightarrow \quad y - 5 = 9(x - 1)$

$\Rightarrow \quad y - 5 = 9x - 9$

$\Rightarrow \quad 9x - y - 9 + 5 = 0$

$\Rightarrow \quad 9x - y - 4 = 0$

which is the required equation of straight line.

Example 28. Find the equation of straight line passes through $(-4, -2)$ and having slope -8.

Sol. Let m be the slope of required line. Therefore $m = -8$. Also it is given that the required line passes through the point $(-4, -2)$. We know that equation of straight line in point slope form is $y - y_1 = m(x - x_1)$.

$\Rightarrow \quad y - (-2) = -8(x - (-4))$

$\Rightarrow \quad y + 2 = -8(x + 4)$

$\Rightarrow \quad y + 2 = -8x - 32$

$\Rightarrow \quad 8x + y + 2 + 32 = 0$

$\Rightarrow \quad 8x + y + 34 = 0$

which is the required equation of straight line.

Example 29. Find the equation of straight line passes through $(0, -8)$ and makes an angle $30°$ with positive direction of X-axis.

Sol. Let m be the slope of required line.

Therefore $m = \tan 30^\circ$

$\Rightarrow \quad m = \frac{1}{\sqrt{3}}$

Also it is given that the required line passes through the point $(0, -8)$.

We know that equation of straight line in point slope form is $y - y_1 = m(x - x_1)$.

$\Rightarrow \quad y - (-8) = \frac{1}{\sqrt{3}}(x - 0)$

$\Rightarrow \quad y + 8 = \frac{x}{\sqrt{3}}$

$\Rightarrow \quad \sqrt{3}\,y + 8\sqrt{3} = x$

$\Rightarrow \quad x - \sqrt{3}\,y - 8\sqrt{3} = 0$

which is the required equation of straight line.

Example 30. Find the equation of straight line passes through $(-9,0)$ and makes an angle $150°$ with positive direction of X-axis.

Sol. Let m be the slope of required line.

Therefore $m = \tan 150^\circ$

$\Rightarrow \quad m = \tan(180^\circ - 30^\circ)$

$\Rightarrow \quad m = -\tan(30^\circ)$

$\Rightarrow \quad m = -\frac{1}{\sqrt{3}}$

Also it is given that the required line passes through the point $(-9,0)$. We know that equation of straight line in point slope form is $y - y_1 = m(x - x_1)$.

$\Rightarrow \quad y - 0 = -\frac{1}{\sqrt{3}}(x - (-9))$

$\Rightarrow \quad -\sqrt{3}\, y = x + 9$

$\Rightarrow \quad x + \sqrt{3}\, y + 9 = 0$

which is the required equation of straight line.

<u>Equation of Straight Line in Two Points form</u>:

Let a non-vertical line passes through two points (x_1, y_1) and (x_2, y_2). $P(x,y)$ be any point on the line (see fig. 3.8), then equation of straight line is $y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$

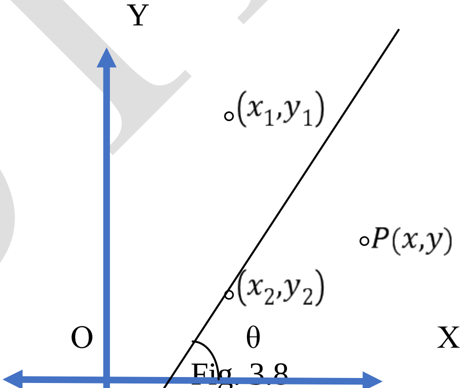


Fig. 3.8

Example 31. Find the equation of straight line passes through the points $(2,-2)$ and $(0,6)$.

Sol. Given that the straight line passes through the points $(2,-2)$ and $(0,6)$.

Comparing these points with (x_1, y_1) and (x_2, y_2) respectively, we get $x_1 = 2$, $y_1 = -2$, $x_2 = 0$ and $y_2 = 6$. We know that equation of straight line in two points slope form is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1).$$

$\Rightarrow \quad y - (-2) = \left(\frac{6 - (-2)}{0 - 2}\right)(x - 2)$

$\Rightarrow \quad y + 2 = \left(\frac{6 + 2}{-2}\right)(x - 2)$

$\Rightarrow \quad y + 2 = -4(x - 2)$

$\Rightarrow \quad y + 2 = -4x + 8$

$\Rightarrow \quad 4x + y - 6 = 0$

which is the required equation of straight line.

Example 32. Find the equation of straight line passes through the points $(0,8)$ and $(5,0)$.

Sol. Given that the straight line passes through the points $(0,8)$ and $(5,0)$.

Comparing these points with (x_1,y_1) and (x_2,y_2) respectively, we get $x_1=0$, $y_1=8$, $x_2=5$ and $y_2=0$. We know that equation of straight line in two points slope form is

$$y-y_1=\left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1).$$

$$\Rightarrow \quad y-8=\left(\frac{0-8}{5-0}\right)(x-0)$$

$$\Rightarrow \quad y-8=\frac{-8x}{5}$$

$$\Rightarrow \quad 5y-40=-8x$$

$$\Rightarrow \quad 8x+5y-40=0$$

which is the required equation of straight line.

Example 33. Find the equation of straight line passes through the points $(7,-4)$ and $(-1,5)$.

Sol. Given that the straight line passes through the points $(7,-4)$ and $(-1,5)$.

Comparing these points with (x_1,y_1) and (x_2,y_2) respectively, we get $x_1=7$, $y_1=-4$, $x_2=-1$ and $y_2=5$. We know that equation of straight line in two points slope form is

$$y-y_1=\left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1).$$

$$\Rightarrow \quad y-(-4)=\left(\frac{5-(-4)}{-1-7}\right)(x-7)$$

$$\Rightarrow \quad y+4=-\frac{9}{8}(x-7)$$

$$\Rightarrow \quad 8y+32=-9x+63$$

$$\Rightarrow \quad 9x+8y-31=0$$

which is the required equation of straight line.

<u>Equation of Straight Line in Slope-Intercept form</u>:

Let a non-vertical line having slope m and its y-intercept is equal to c. $P(x,y)$ be any point on the line (see Fig. 3.9), then equation of straight line is $y=mx+c$

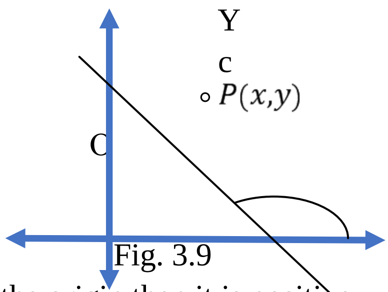


Fig. 3.9

<u>Note</u>: **(i)** If intercept c is given above the X-axis or above the origin then it is positive.
(ii) If intercept c is given below the X-axis or below the origin then it is negative.

Example 34. Find the equation of straight line having slope 3 and cuts of an intercept -2 on Y-axis.

Sol. Given that the slope m of straight line is 3 and Y-intercept is -2 i.e. $c=-2$. We know that equation of straight line in slope-intercept form is

$$y = mx + c$$

$\Rightarrow \quad y = 3x - 2$

$\Rightarrow \quad 3x - y - 2 = 0$

which is the required equation of straight line.

Example 35. Find the equation of straight line having slope -6 and cuts of an intercept 5 on Y-axis above the origin.

Sol. Given that the slope m of straight line is -6 and Y-intercept is 5 i.e. $c = 5$.
c is taken positive as Y-intercept is above the origin.
We know that equation of straight line in slope-intercept form is

$$y = mx + c$$

$\Rightarrow \quad y = -6x + 5$

$\Rightarrow \quad 6x + y - 5 = 0$

which is the required equation of straight line.

Example 36. Find the equation of straight line having slope 2 and cuts of an intercept 9 on Y-axis below the origin.

Sol. Given that the slope m of straight line is 2 and Y-intercept is -9 i.e. $c = -9$. c is taken negative as Y-intercept is below the origin. We know that equation of straight line in slope-intercept form is

$$y = mx + c$$

$\Rightarrow \quad y = 2x - 9$

$\Rightarrow \quad 2x - y - 9 = 0$

which is the required equation of straight line.

Example 37. Find the equation of straight line which makes an angle $45°$ with X-axis and cuts of an intercept 8 on Y-axis below the X-axis.

Sol. Given that the required line makes an angle $45°$ with X-axis.
Therefore slope m of straight line is given by $m = \tan 45^\circ$ *i.e.* $m = 1$.
Also Y-intercept is -8 i.e. $c = -8$. c is taken negative as Y-intercept is below the X-axis. We know that equation of straight line in slope-intercept form is

$$y = mx + c$$

$\Rightarrow \quad y = 1x - 8$

$\Rightarrow \quad x - y - 8 = 0$

which is the required equation of straight line.

Example 38. Find the equation of straight line which makes an angle $60°$ with X-axis and cuts of an intercept 5 on Y-axis above the X-axis.

Sol. Given that the required line makes an angle $60°$ with X-axis.
Therefore slope m of straight line is given by $m = \tan 60^\circ$ *i.e.* $m = \sqrt{3}$.
Also Y-intercept is 5 i.e. $c = 5$. c is taken positive as Y-intercept is above the X-axis. We know that equation of straight line in slope-intercept form is

$$y = mx + c$$

$\Rightarrow \quad y = \sqrt{3}\,x + 5$

$\Rightarrow \quad \sqrt{3}\,x - y + 5 = 0$

which is the required equation of straight line.

Example 39. Find the equation of straight line which passes through the points $(0,3)$ and $(2,0)$ and cuts of an intercept 12 on Y-axis below the origin.

Sol. Given that the required line passes through the points $(0,3)$ and $(2,0)$. Therefore slope m of straight line is given by $m=\frac{0-3}{2-0}$ *i.e.* $m=-\frac{3}{2}$. Also Y-intercept is -12 i.e. $c=-12$. c is taken negative as Y-intercept is below the origin. We know that equation of straight line in slope-intercept form is

$$y=mx+c$$

$$\Rightarrow \quad y=-\frac{3}{2}x-12$$

$$\Rightarrow \quad 2y=-3x-24$$

$$\Rightarrow \quad 3x+2y+24=0$$

which is the required equation of straight line.

Equation of Straight Line in Intercept form:

Let a non-vertical line having intercepts a and b on X-axis and Y-axis respectively. $P(x,y)$ be any point on the line (Fig. 3.10), then equation of straight line is

$$\frac{x}{a}+\frac{y}{b}=1$$

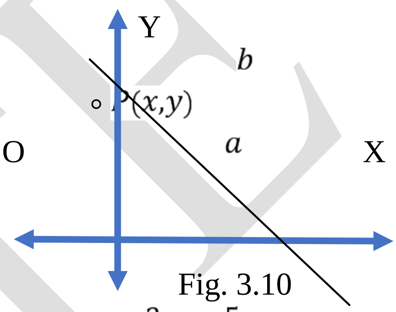


Fig. 3.10

Example 40. Find the equation of straight line which makes intercepts 2 and 5 on X-axis and Y-axis respectively.

Sol. Given that X-intercept is 2 and Y-intercept is 5

i.e. $a=2$ and $b=5$

We know that equation of straight line in Intercept form is

$$\frac{x}{a}+\frac{y}{b}=1$$

$$\Rightarrow \quad \frac{x}{2}+\frac{y}{5}=1$$

$$\Rightarrow \quad \frac{5x+2y}{10}=1$$

$$\Rightarrow \quad 5x+2y=10$$

$$\Rightarrow \quad 5x+2y-10=0$$

which is the required equation of straight line.

Example 41. Find the equation of straight line which makes intercepts 3 and -15 on the axes.

Sol. Given that X-intercept is 3 and Y-intercept is -15

i.e. $a=3$ and $b=-15$

We know that equation of straight line in Intercept form is

$$\frac{x}{a}+\frac{y}{b}=1$$

$$\Rightarrow \quad \frac{x}{3}+\frac{y}{-15}=1$$

$$\Rightarrow \quad \frac{x}{3}-\frac{y}{15}=1$$

$$\Rightarrow \quad \frac{5x-y}{15}=1$$

$\Rightarrow \quad 5x - y = 15$

$\Rightarrow \quad 5x - y - 15 = 0$

which is the required equation of straight line.

Example 42. Find the equation of straight line which passes through $(1, -4)$ and makes intercepts on axes which are equal in magnitude and opposite in sign.

Sol. Let the intercepts on the axes are p and $-p$

i.e. $a = p$ and $b = -p$

We know that equation of straight line in Intercept form is

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\Rightarrow \quad \frac{x}{p} + \frac{y}{-p} = 1$$

$$\Rightarrow \quad \frac{x}{p} - \frac{y}{p} = 1$$

$$\Rightarrow \quad \frac{x-y}{p} = 1$$

$$\Rightarrow \quad x - y = p \qquad (1)$$

Given that this line passes through $(1, -4)$.

Therefore put $x = 1$ and $y = -4$ in (1), we get

$$1 - (-4) = p$$

$$\Rightarrow \quad p = 5$$

Using this value in (1), we get

$$x - y = 5$$

$$\Rightarrow \quad x - y - 5 = 0$$

which is the required equation of straight line.

Example 43. Find the equation of straight line which passes through $(1,4)$ and sum of whose intercepts on axes is 10.

Sol. Let the intercepts on the axes are p and $10 - p$

i.e. $a = p$ and $b = 10 - p$

We know that equation of straight line in Intercept form is

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\Rightarrow \quad \frac{x}{p} + \frac{y}{10-p} = 1$$

$$\Rightarrow \quad \frac{(10-p)x + py}{p(10-p)} = 1$$

$$\Rightarrow \quad (10-p)x + py = p(10-p) \qquad (1)$$

Given that this line passes through $(1,4)$. Therefore put $x = 1$ and $y = 4$ in (1), we get

$$(10-p)(1) + p(4) = p(10-p)$$

$$\Rightarrow \quad 10 - p + 4p = 10p - p^2$$

$$\Rightarrow \quad p^2 - 7p + 10 = 0$$

$$\Rightarrow \quad p^2 - 5p - 2p + 10 = 0$$

$$\Rightarrow \quad p(p-5) - 2(p-5) = 0$$

$$\Rightarrow \quad (p-2)(p-5) = 0$$

$$\textit{either} \quad p-2=0 \quad \textit{or} \quad p-5=0$$
$$\textit{either} \quad p=2 \quad \textit{or} \quad p=5$$

Put $p=2$ in (1), we get

$\Rightarrow \quad (10-2)x+2y=2(10-2)$

$\Rightarrow \quad 8x+2y=16$

$\Rightarrow \quad 4x+y=8 \qquad (2)$

Put $p=5$ in (1), we get

$\Rightarrow \quad (10-5)x+5y=5(10-5)$

$\Rightarrow \quad 5x+5y=25$

$\Rightarrow \quad x+y=5 \qquad (3)$

(2) and (3) are the required equations of straight lines.

Equation of Straight Line in Normal form:

Let p be the length of perpendicular from the origin to the straight line and α be the angle which this perpendicular makes with the positive direction of X-axis. (x,y) be any point on the line (see Fig.p 3.11), then equation of straight line is
X

$$p = x\ cos\alpha + y\ sin\alpha$$

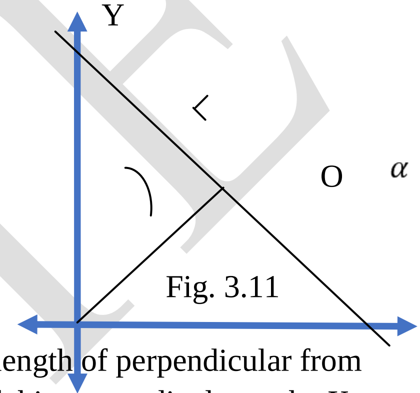


Fig. 3.11

Example 44. Find the equation of straight line such that the length of perpendicular from the origin to the straight line is 2 and the inclination of this perpendicular to the X-axis is $120°$.

Sol. We know that equation of straight line in Normal form is

$$x\cos\alpha + y\sin\alpha = p \qquad (1)$$

where p be the length of perpendicular from the origin to the straight line and α be the angle which this perpendicular makes with the positive direction of X-axis.

Here $p=2$ and $\alpha=120^\circ$. Putting these values in (1), we get

$$x\cos 120^\circ + y\sin 120^\circ = 2$$

$\Rightarrow \quad x\cos(180^\circ-60^\circ)+y\sin(180^\circ-60^\circ)=2$

$\Rightarrow \quad -x\cos(60^\circ)+y\sin(60^\circ)=2$

$\Rightarrow \quad -x\left(\frac{1}{2}\right)+y\left(\frac{\sqrt{3}}{2}\right)=2$

$\Rightarrow \quad \frac{-x+\sqrt{3}y}{2}=2$

$\Rightarrow \quad -x+\sqrt{3}y=4$

$\Rightarrow \quad -x+\sqrt{3}y-4=0$

which is the required equation of straight line.

Example 45. Find the equation of straight line such that the length of perpendicular from the origin to the straight line is 7 and the inclination of this perpendicular to the X-axis is $45°$.

Sol. We know that equation of straight line in Normal form is

$$x\cos\alpha + y\sin\alpha = p \qquad (1)$$

where p be the length of perpendicular from the origin to the straight line and α be the angle which this perpendicular makes with the positive direction of X-axis.

Here $p=7$ and $\alpha=45^\circ$. Put these values in (1), we get

$$x\cos 45^\circ + y\sin 45^\circ = 7$$

$$\Rightarrow \quad x\left(\frac{1}{\sqrt{2}}\right)+y\left(\frac{1}{\sqrt{2}}\right)=7$$

$$\Rightarrow \quad x+y=7\sqrt{2}$$

$$\Rightarrow \quad x+y-7\sqrt{2}=0$$

which is the required equation of straight line.

<u>Angle Between Two Straight Lines</u>:

Two intersecting lines always intersects at two angles in which one angle is acute angle and other angle is obtuse angle. The sum of both the angles is 180° i.e. they are supplementary to each other. For Ex, if one angle between intersecting lines is 60° then other angle is $180^\circ-60^\circ=120^\circ$. Generally, we take acute angle as the angle between the lines (see Fig. 3.12).

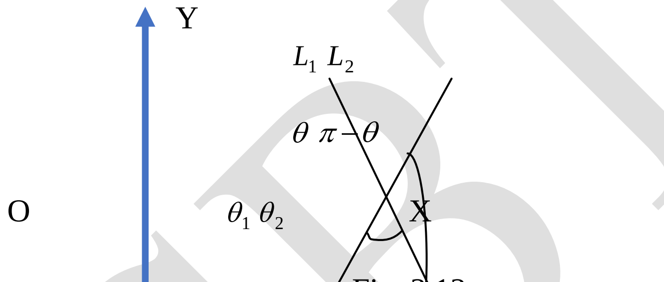


Fig. 3.12

Let $L_1 \,\&\, L_2$ be straight lines and $m_1 \,\&\, m_2$ be their slopes respectively. Also, let $\theta_1 \,\&\, \theta_2$ be the angles which $L_1 \,\&\, L_2$ make with positive X-axis respectively.

Therefore $m_1=\tan(\theta_1) \,\&\, m_2=\tan(\theta_2)$. Let θ be the acute angle between lines, then

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1\,m_2}\right| \qquad or \qquad \tan\theta=\left|\frac{m_2-m_1}{1+m_2\,m_1}\right|$$

Example 46. Find the acute angle between the lines whose slopes are 1 and 0.

Sol. Given that slopes of lines are 1 and 0.

Let $m_1=1$ and $m_2=0$.

Also let θ be the acute angle between lines. Therefore, $\tan\theta=\left|\frac{m_1-m_2}{1+m_1\,m_2}\right|$

$$\Rightarrow \quad \tan\theta=\left|\frac{1-0}{1+(1)(0)}\right|$$

$$\Rightarrow \quad \tan\theta=\left|\frac{1}{1+0}\right| = 1$$

$$\Rightarrow \quad \tan\theta=\tan\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \quad \theta=\frac{\pi}{4}$$

which is the required acute angle.

Example 47. Find the acute angle between the lines whose slopes are $2+\sqrt{3}$ and $2-\sqrt{3}$.

Sol. Given that slopes of lines are $2+\sqrt{3}$ and $2-\sqrt{3}$.

Let $m_1 = 2+\sqrt{3}$ and $m_2 = 2-\sqrt{3}$.

Also let θ be the acute angle between lines.

Therefore, $\tan\theta = \left|\dfrac{m_1 - m_2}{1+m_1 m_2}\right|$

$$\Rightarrow \qquad \tan\theta = \left|\frac{\left(2+\sqrt{3}\right)-\left(2-\sqrt{3}\right)}{1+\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{2+\sqrt{3}-2+\sqrt{3}}{1+(4-3)}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{2\sqrt{3}}{2}\right| = \sqrt{3}$$

$$\Rightarrow \qquad \tan\theta = \tan\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \qquad \theta = \frac{\pi}{3}$$

which is the required acute angle.

Example 48. Find the obtuse angle between the lines whose slopes are $\sqrt{3}$ and $\frac{1}{\sqrt{3}}$.

Sol. Given that slopes of lines are $\sqrt{3}$ and $\frac{1}{\sqrt{3}}$. Let $m_1 = \sqrt{3}$ and $m_2 = \dfrac{1}{\sqrt{3}}$.

Also let θ be the acute angle between lines. Therefore, $\tan\theta = \left|\dfrac{m_1 - m_2}{1+m_1 m_2}\right|$

$$\Rightarrow \qquad \tan\theta = \left|\frac{\sqrt{3}-\dfrac{1}{\sqrt{3}}}{1+\left(\sqrt{3}\right)\left(\dfrac{1}{\sqrt{3}}\right)}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{\dfrac{3-1}{\sqrt{3}}}{1+1}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{2}{2\sqrt{3}}\right|$$

$$\Rightarrow \qquad \tan\theta = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \qquad \tan\theta = \tan\left(30^\circ\right)$$

$$\Rightarrow \qquad \theta = 30^\circ$$

Therefore, $180^\circ - \theta$ is the obtuse angle between the lines.

i.e. $180^\circ - 30^\circ = 150^\circ$ is the obtuse angle between the lines.

Example 49. Find the angle between the lines whose slopes are -3 and 5.

Sol. Given that slopes of lines are -3 and 5. Let $m_1 = -3$ and $m_2 = 5$.

Also let θ be the angle between lines. Therefore, $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\Rightarrow \qquad \tan\theta = \left|\frac{-3-5}{1+(-3)(5)}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{-8}{1-15}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{-8}{-14}\right|$$

$$\Rightarrow \qquad \tan\theta = \frac{4}{7}$$

$$\Rightarrow \qquad \theta = \tan^{-1}\left(\frac{4}{7}\right)$$

which is the required angle.

Example 50. Find the angle between the lines joining the points $(0,0)$, $(2,3)$ and $(2,-2)$, $(3,5)$.

Sol. Let m_1 be the slope of the line joining $(0,0)$ and $(2,3)$ and m_2 be the slope of the line joining $(2,-2)$ and $(3,5)$. Then

$$\Rightarrow \qquad m_1 = \frac{3-0}{2-0} = \frac{3}{2}$$

and $m_2 = \frac{5-(-2)}{3-2} = \frac{7}{1} = 7$.

Also let θ be the angle between lines.

Therefore, $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\Rightarrow \qquad \tan\theta = \left|\frac{\frac{3}{2} - 7}{1 + \left(\frac{3}{2}\right)(7)}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{\frac{3-14}{2}}{\frac{2+21}{2}}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{\frac{-11}{2}}{\frac{23}{2}}\right|$$

$$\Rightarrow \qquad \tan\theta = \frac{11}{23}$$

$$\Rightarrow \qquad \theta = \tan^{-1}\left(\frac{11}{23}\right)$$

which is the required angle.

Example 51. Find the angle between the lines joining the points $(6,-5)$, $(-2,1)$ and $(0,3)$, $(-8,6)$.

Sol. Let m_1 be the slope of the line joining $(6,-5)$ and $(-2,1)$ and m_2 be the slope of the line joining $(0,3)$ and $(-8,6)$. Then

$$\Rightarrow \quad m_1=\frac{1-(-5)}{-2-6}=-\frac{6}{8}=-\frac{3}{4}$$

and $m_2=\frac{6-3}{-8-0}=-\frac{3}{8}$.

Also let θ be the angle between lines. Therefore, $\tan\theta=\left|\frac{m_1-m_2}{1+m_1 m_2}\right|$

$$\Rightarrow \quad \tan\theta=\left|\frac{-\frac{3}{4}-\left(-\frac{3}{8}\right)}{1+\left(-\frac{3}{4}\right)\left(-\frac{3}{8}\right)}\right|$$

$$\Rightarrow \quad \tan\theta=\left|\frac{-\frac{3}{4}+\frac{3}{8}}{1+\frac{9}{32}}\right|$$

$$\Rightarrow \quad \tan\theta=\left|\frac{\frac{-6+3}{8}}{\frac{32+9}{32}}\right|$$

$$\Rightarrow \quad \tan\theta=\left|\frac{\frac{-3}{8}}{\frac{41}{32}}\right|$$

$$\Rightarrow \quad \tan\theta=\left|\frac{-3}{8}\times\frac{32}{41}\right|$$

$$\Rightarrow \quad \tan\theta=\frac{12}{41}$$

$$\Rightarrow \quad \theta=\tan^{-1}\left(\frac{12}{41}\right)$$

which is the required angle.

Example 52. Find the angle between the pair of straight lines $(-2+\sqrt{3})x+y+9=0$ and $(2+\sqrt{3})x-y+20=0$.

Sol. Given equations of lines

$$(-2+\sqrt{3})x+y+9=0 \qquad (1)$$

and $$(2+\sqrt{3})x-y+20=0 \qquad (2)$$

Let m_1 be the slope of the line (1) and m_2 be the slope of the line (2).

$$\Rightarrow \quad m_1=-\frac{-2+\sqrt{3}}{1}=2-\sqrt{3}$$

and $m_2 = -\frac{2+\sqrt{3}}{-1} = 2+\sqrt{3}$.

Also let θ be the angle between lines.

Therefore, $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\Rightarrow \qquad \tan\theta = \left|\frac{\left(2-\sqrt{3}\right)-\left(2+\sqrt{3}\right)}{1+\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{2-\sqrt{3}-2-\sqrt{3}}{1+4-3}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{-2\sqrt{3}}{2}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|-\sqrt{3}\right|$$

$$\Rightarrow \qquad \tan\theta = \sqrt{3}$$

$$\Rightarrow \qquad \tan\theta = \tan\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \qquad \theta = \frac{\pi}{3}$$

which is the required angle.

Example 53. Find the angle between the pair of straight lines $x+\sqrt{3}\,y-8=0$ and $x-\sqrt{3}\,y+2=0$.

Sol. Given equations of lines

$x+\sqrt{3}\,y-8=0$ (1)

and $x-\sqrt{3}\,y+2=0$ (2)

Let m_1 be the slope of the line (1) and m_2 be the slope of the line (2).

$$\Rightarrow \qquad m_1 = -\frac{1}{\sqrt{3}}$$

and $m_2 = -\frac{1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$.

Also let θ be the angle between lines. Therefore, $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\Rightarrow \qquad \tan\theta = \left|\frac{-\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}}{1+\left(-\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{-\frac{2}{\sqrt{3}}}{1-\frac{1}{3}}\right|$$

$$\Rightarrow \qquad \tan\theta = \left| \frac{-\frac{2}{\sqrt{3}}}{\frac{3-1}{3}} \right|$$

$$\Rightarrow \qquad \tan\theta = \left| \frac{-\frac{2}{\sqrt{3}}}{\frac{2}{3}} \right|$$

$$\Rightarrow \qquad \tan\theta = \left| -\frac{3}{\sqrt{3}} \right|$$

$$\Rightarrow \qquad \tan\theta = \frac{3}{\sqrt{3}} = \frac{3\sqrt{3}}{\sqrt{3}\times\sqrt{3}} = \frac{3\sqrt{3}}{3}$$

$$\Rightarrow \qquad \tan\theta = \sqrt{3}$$

$$\Rightarrow \qquad \tan\theta = \tan\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \qquad \theta = \frac{\pi}{3}$$

which is the required angle.

Example 54. Find the equations of straight lines making an angle $45°$ with the line $6x + 5y - 1 = 0$ and passing through the point $(2, -1)$.

Sol. Given that equations of lines are

$$6x + 5y - 1 = 0 \qquad (1)$$

Let m_1 be the slope of the line (1)

$$\Rightarrow \qquad m_1 = -\frac{6}{5}.$$

Let m_2 be the slope of required line. Therefore, $\tan 45^\circ = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$

$$\Rightarrow \qquad 1 = \left| \frac{-\frac{6}{5} - m_2}{1 + \left(-\frac{6}{5}\right)(m_2)} \right|$$

$$\Rightarrow \qquad 1 = \left| \frac{\frac{-6-5m_2}{5}}{\frac{5-6m_2}{5}} \right|$$

$$\Rightarrow \qquad 1 = \left| \frac{-6-5m_2}{5-6m_2} \right|$$

$$\Rightarrow \qquad 1 = \pm\left(\frac{-6-5m_2}{5-6m_2} \right)$$

$$\Rightarrow \qquad 5 - 6m_2 = \pm(-6 - 5m_2)$$

Taking positive sign, we get

$$5-6m_2=+(-6-5m_2)$$

$\Rightarrow \quad 5-6m_2=-6-5m_2$

$\Rightarrow \quad -m_2=-11$

$\Rightarrow \quad m_2=11$

So, equation of line passing through $(2,-1)$ with slope 11 is

$$y+1=11(x-2)$$

$\Rightarrow \quad y+1=11x-22$

$\Rightarrow \quad 11x-y-23=0 \qquad (2)$

Now taking negative sign, we get

$$5-6m_2=-(-6-5m_2)$$

$\Rightarrow \quad 5-6m_2=6+5m_2$

$\Rightarrow \quad -11m_2=1$

$\Rightarrow \quad m_2=-\frac{1}{11}$

So, equation of line passing through $(2,-1)$ with slope $-\frac{1}{11}$ is

$$y+1=-\frac{1}{11}(x-2)$$

$\Rightarrow \quad 11y+11=-x+2$

$\Rightarrow \quad x+11y+9=0 \qquad (3)$

(2) and (3) are required equations of straight lines.

EXERCISE – II

1. The slope of Y-axis is:

(a) Infinite (b) 0 (c) 1/2 (d) 1

2. If two lines are intersecting at an angle of 60^0 then, the other angle between these two lines is:

(a) 120^0 (b) 60^0 (c) 90^0 (d) 180^0

3. If the equation of straight line is ax + by + c = 0, then slope of straight line is:

(a) $-\frac{b}{a}$ (b) $-\frac{a}{b}$ (c) $\frac{b}{a}$ (d) c

4. The equation of a straight line passing through (x_1, y_1) and having slope m is:

(a) $y - y_1 = m (x - x_1)$ (b) $x - x_1 = m (y - y_1)$

(c) $y - y_1 = - m (x - x1)$ (d) None of these

5. Find the straight line which passes through the following pairs of points.

(i) (-11, -5), (-3, 10) (ii) (0, 0), (10, -12)

6. Find the equation of straight line which makes an angle 60^0 with x-axis and cuts on intercepts 5 on y-axis above the x-axis.
7. Find the equation of straight line which passes through (2, 3) and makes equal intercepts in sign and magnitude on axes.
8. Find the equation of straight line passes through (2, 4) and sum of whose intercepts on axes is 15.

9. Find the equation of straight line such that the length of perpendicular from the origin to the straight line is 10 and the inclination of this perpendicular to the x-axis is 60^0.

10. Find the angle between the lines joining the points (0, 0), (4, 6) and (1, -1), (6, 10)

11. Find the angle between the pair of straight lines $(-4+\sqrt{3})$ x + y + 9 = 0 and $(4+\sqrt{3})$ x - y + 10 = 0.

12. Find the equation of straight line making an angle 60^0 with the line 6x + 5y -1 = 0 and passing through the point (1, -1).

ANSWERS

1. (a) 2. (a) 3. (b) 4. (a) 5. (i) 15x-8y+125=0 (ii) 6x+15y = 0 6. $y=\sqrt{3}x+5$

7. .x+y=5 8. $\frac{x}{10}+\frac{y}{5}=1$; $\frac{x}{3}+\frac{y}{12}=1$ 9. $\frac{x}{2}+\frac{\sqrt{3}}{2}y=10$ 10. $\theta=\tan^{-1}(7/43)$

11. $\theta=\tan^{-1}\left(-\sqrt{3}/7\right)$ 12. $y+1=\left(\frac{5\sqrt{3}-6}{6\sqrt{3}+5}\right)(x-2)$; $y+1=\left(\frac{5\sqrt{3}+6}{6\sqrt{3}-5}\right)(x-2)$

3.3 CIRCLE

Circle: Circle is the locus of a point which moves in a plane such that its distance from a fixed point always remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

In figure 3.13, $C(h,k)$ be the centre of the circle, r be the radius of the circle and $P(x,y)$ be the moving point on the circumference of the circle.

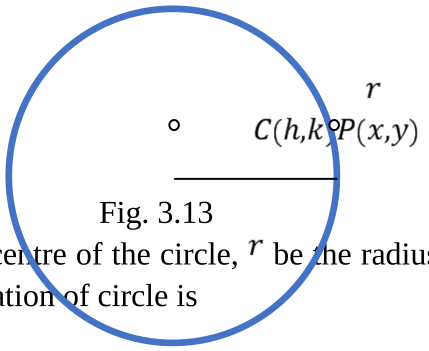


Fig. 3.13

Standard form of Equation of Circle: Let $C(h,k)$ be the centre of the circle, r be the radius of the circle and $P(x,y)$ be any point on the circle, then equation of circle is

$$(x-h)^2+(y-k)^2=r^2 \qquad (1)$$

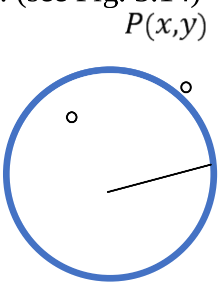
which is known as standard form of equation of circle. This also known as central form of equation of circle.

Some Particular Cases:

Let $C(h,k)$ be the centre of the circle, r be the radius of the circle and $P(x,y)$ be any point on the circle:

(i) When the centre of the circle coincides with the origin i.e. $h=k=0$: (see Fig. 3.14)
Thus equation (1) becomes:

$$\Rightarrow \qquad (x-0)^2+(y-0)^2=r^2$$



$$\Rightarrow \qquad x^2+y^2=r^2$$

$C(0,0)$

Fig. 3.14

(ii) When the circle passes through the origin: (see figure 3.15)

Let CR be the perpendicular from the centre on X-axis. Therefore,

$$OR^2+CR^2=OC^2$$

$$\Rightarrow \qquad (h-0)^2+(k-0)^2=r^2$$

$$\Rightarrow \qquad h^2+k^2=r^2$$

Thus equation (1) becomes:

$$\Rightarrow \qquad (x-h)^2+(y-k)^2=h^2+k^2$$

$$\Rightarrow \qquad x^2+h^2-2hx+y^2+k^2-2ky=h^2+k^2$$

$$\Rightarrow \qquad x^2+y^2-2hx-2ky=0$$

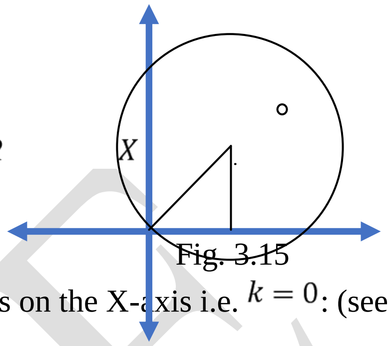


Fig. 3.15

(iii) When the circle passes through the origin and centre lies on the X-axis i.e. $k=0$: (see figure 3.16)

In this case radius $r=|h|$ Thus equation (1) becomes:

$$\Rightarrow \qquad (x-h)^2+(y-0)^2=h^2$$

$$\Rightarrow \qquad x^2+h^2-2hx+y^2=h^2$$

$$\Rightarrow \qquad x^2+y^2-2hx=0$$

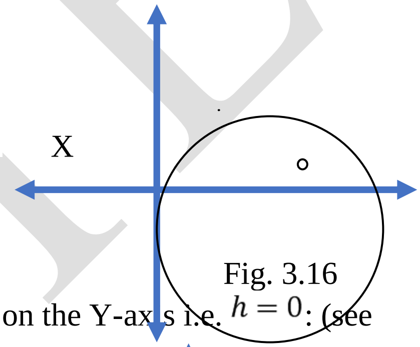


Fig. 3.16

(iv) When the circle passes through the origin and centre lies on the Y-axis i.e. $h=0$: (see Fig. 3.17)

In this case radius $r=|k|$ Thus equation (1) becomes:

$$\Rightarrow \qquad (x-0)^2+(y-k)^2=k^2$$

$$\Rightarrow \qquad x^2+y^2+k^2-2ky=k^2$$

$$\Rightarrow \qquad x^2+y^2-2ky=0$$

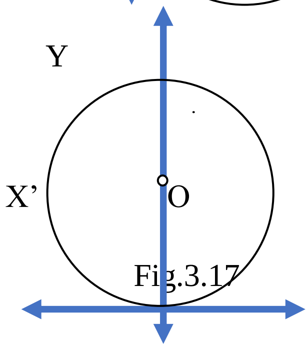


Fig.3.17

(v) When the circle touches the X-axis: (see figure 3.18)

In this case radius $r=|k|$

Thus equation (1) becomes: $\Rightarrow \qquad (x-h)^2+(y-k)^2=k^2$

$$\Rightarrow \qquad x^2+h^2-2hx+y^2+k^2-2ky=k^2$$

$$\Rightarrow \qquad x^2+y^2-2hx-2ky+h^2=0$$

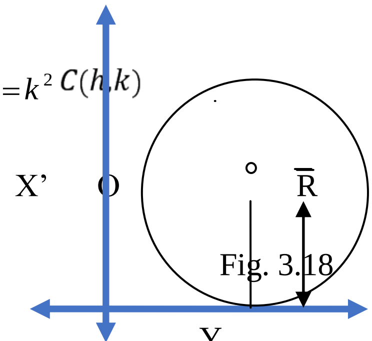


Fig. 3.18

(vi) When the circle touches the Y-axis: (see figure. 3.19)

In this case radius $r=|h|$ Thus equation (1) becomes:

$$\Rightarrow \qquad (x-h)^2+(y-k)^2=h^2$$

$$\Rightarrow \qquad x^2+h^2-2hx+y^2+k^2-2ky=h^2$$

$$\Rightarrow \qquad x^2+y^2-2hx-2ky+k^2=0$$

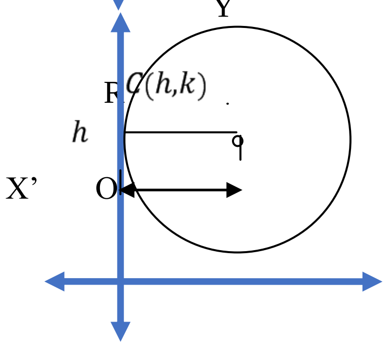


Fig. 3.19

(vii) When the circle touches both the axes: (see figure 3.20)

In this case radius $r = |h| = |k|$ Thus equation (1) becomes:

$$\Rightarrow \quad (x-h)^2 + (y-h)^2 = h^2$$
$$\Rightarrow \quad x^2 + h^2 - 2hx + y^2 + h^2 - 2hy = h^2$$
$$\Rightarrow \quad x^2 + y^2 - 2hx - 2hy + h^2 = 0$$

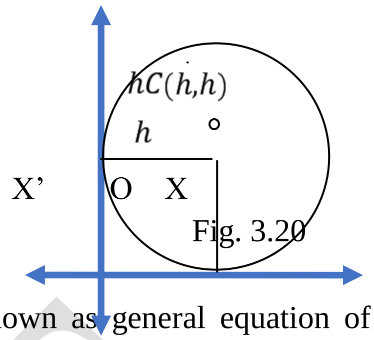


Fig. 3.20

General Equation of Circle:

An equation of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ is known as general equation of circle, where g , f and c are arbitrary constants.

To convert general equation of circle into standard equation:

Let the general equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (2)$$
$$\Rightarrow \quad (x^2 + 2gx) + (y^2 + 2fy) + c = 0$$
$$\Rightarrow \quad (x^2 + 2gx + g^2 - g^2) + (y^2 + 2fy + f^2 - f^2) + c = 0$$
$$\Rightarrow \quad (x+g)^2 - g^2 + (y+f)^2 - f^2 + c = 0$$
$$\Rightarrow \quad (x+g)^2 + (y+f)^2 = f^2 + g^2 - c$$
$$\Rightarrow \quad (x+g)^2 + (y+f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$

which is the required standard form. Comparing it with $(x-h)^2 + (y-k)^2 = r^2$, we get $h = -g$, $k = -f$ and $r = \sqrt{g^2 + f^2 - c}$.

Hence, centre of given circle (2) is $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$. We observe that the centre of circle (2) is $\left(-\frac{1}{2} \times Coefficient\ of\ x, -\frac{1}{2} \times Coefficient\ of\ y\right)$.

Equation of Circle in Diametric Form:

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of diameter of a circle then the equation of circle is given by

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Example 55 Find the centre and radius of the following circles:

(i) $x^2 + y^2 + 2x + 4y - 4 = 0$ (ii) $x^2 + y^2 - 6x + 10y + 3 = 0$

(iii) $x^2 + y^2 - 3x - 5y - 1 = 0$ (iv) $2x^2 + 2y^2 + 5x - 6y + 2 = 0$

(v) $3x^2 + 3y^2 - 6x - 15y + 12 = 0$ (vi) $x^2 + y^2 - 12y + 6 = 0$

(vii) $x^2 + y^2 + 10x - 3 = 0$ (viii) $x^2 + y^2 + 7x - 9y = 0$

Sol.

(i) Given that equation of circle is

$x^2 + y^2 + 2x + 4y - 4 = 0$ (1)

Compare (1) with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$2g = 2,\ 2f = 4$ and $c = -4$

i.e. $g=1$, $f=2$ and $c=-4$.

We know that centre of circle is given by $(-g,-f)$ and radius r is given by $\sqrt{g^2+f^2-c}$

Therefore, centre of circle (1) is $(-1,-2)$ and radius r of circle (1) is

$$r=\sqrt{1^2+2^2-(-4)}$$

$$\Rightarrow \quad r=\sqrt{1+4+4}$$

$$\Rightarrow \quad r=\sqrt{9}=3$$

(ii) Given that equation of circle is

$$x^2+y^2-6x+10y+3=0 \qquad (2)$$

Compare (2) with $x^2+y^2+2gx+2fy+c=0$, we get

$2g=-6$, $2f=10$ and $c=3$ *i.e.* $g=-3$, $f=5$ and $c=3$.

We know that centre of circle is given by $(-g,-f)$ and radius r is given by $\sqrt{g^2+f^2-c}$. Therefore, centre of circle (2) is $(3,-5)$ and radius r of circle (2) is

$$r=\sqrt{(-3)^2+5^2-3}$$

$$\Rightarrow \quad r=\sqrt{9+25-3}$$

$$\Rightarrow \quad r=\sqrt{31}$$

(iii) Given that equation of circle is

$$x^2+y^2-3x-5y-1=0 \qquad (3)$$

Compare (3) with $x^2+y^2+2gx+2fy+c=0$, we get

$2g=-3$, $2f=-5$ and $c=-1$

i.e. $g=-\frac{3}{2}$, $f=-\frac{5}{2}$ and $c=-1$.

We know that centre of circle is given by $(-g,-f)$ and radius r is given by $\sqrt{g^2+f^2-c}$ Therefore, centre of circle (3) is $\left(\frac{3}{2},\frac{5}{2}\right)$ and radius r of circle (3) is

$$r=\sqrt{\left(-\frac{3}{2}\right)^2+\left(-\frac{5}{2}\right)^2-(-1)}$$

$$\Rightarrow \quad r=\sqrt{\frac{9}{4}+\frac{25}{4}+1}$$

$$\Rightarrow \quad r=\sqrt{\frac{9+25+4}{4}}=\sqrt{\frac{38}{4}}$$

$$\Rightarrow \quad r=\sqrt{\frac{19}{2}}$$

(iv) Given that equation of circle is

$$2x^2+2y^2+5x-6y+2=0$$

Diving this equation by 2, we get

$$x^2+y^2+\tfrac{5}{2}x-3y+1=0 \qquad (4)$$

Compare (4) with $x^2+y^2+2gx+2fy+c=0$, we get

$2g=\frac{5}{2}$, $2f=-3$ and $c=1$

i.e. $g=\frac{5}{4}$, $f=-\frac{3}{2}$ and $c=1$.

We know that centre of circle is given by $(-g,-f)$ and radius r is given by $\sqrt{g^2+f^2-c}$ Therefore, centre of circle (4) is $\left(-\frac{5}{4},\frac{3}{2}\right)$ and radius r of circle (4) is

$$r=\sqrt{\left(\frac{5}{4}\right)^2+\left(-\frac{3}{2}\right)^2-1}$$

$$\Rightarrow \quad r=\sqrt{\frac{25}{16}+\frac{9}{4}-1}$$

$$\Rightarrow \quad r=\sqrt{\frac{25+36-16}{16}}=\sqrt{\frac{45}{16}}$$

$$\Rightarrow \quad r=\frac{3\sqrt{5}}{4}$$

(v) Given that equation of circle is

$3x^2+3y^2-6x-15y+12=0$

Diving this equation by 3, we get

$x^2+\ y^2-2x-5y+4=0$ (5)

Compare (5) with $x^2+y^2+2gx+2fy+c=0$, we get

$2g=-2$, $2f=-5$ and $c=4$

i.e. $g=-1$, $f=-\frac{5}{2}$ and $c=4$.

We know that centre of circle is given by $(-g,-f)$ and radius r is given by $\sqrt{g^2+f^2-c}$ Therefore, centre of circle (5) is $\left(1,\frac{5}{2}\right)$ and radius r of circle (5) is

$$r=\sqrt{(-1)^2+\left(-\frac{5}{2}\right)^2-4}$$

$$\Rightarrow \quad r=\sqrt{1+\frac{25}{4}-4}$$

$$\Rightarrow \quad r=\sqrt{\frac{4+25-16}{4}}=\sqrt{\frac{13}{4}}$$

$$\Rightarrow \quad r=\frac{\sqrt{13}}{2}$$

(vi) Given that equation of circle is

$x^2+y^2-12y+6=0$ (6)

Compare (6) with $x^2+y^2+2gx+2fy+c=0$, we get

$2g=0$, $2f=-12$ and $c=6$

i.e. $g=0$, $f=-6$ and $c=6$. We know that centre of circle is given by $(-g,-f)$ and radius r is given by $\sqrt{g^2+f^2-c}$.

Therefore, centre of circle (6) is $(0,6)$ and radius r of circle (6) is

$$r=\sqrt{(0)^2+(-6)^2-6}$$

$$\Rightarrow \quad r=\sqrt{0+36-6}$$

$$\Rightarrow \quad r=\sqrt{30}$$

(vii) Given that equation of circle is

$$x^2+y^2+10x-3=0 \tag{7}$$

Compare (7) with $x^2+y^2+2gx+2fy+c=0$, we get

$2g=10$, $2f=0$ and $c=-3$

i.e. $g=5$, $f=0$ and $c=-3$.

We know that centre of circle is given by $(-g,-f)$ and radius r is given by $\sqrt{g^2+f^2-c}$ Therefore, centre of circle (7) is $(-5,0)$ and radius r of circle (7) is

$$r=\sqrt{(5)^2+(0)^2-(-3)}$$

$$\Rightarrow \quad r=\sqrt{25+0+3}=\sqrt{28}$$

$$\Rightarrow \quad r=2\sqrt{7}$$

(viii) Given that equation of circle is

$$x^2+y^2+7x-9y=0 \tag{8}$$

Compare (8) with $x^2+y^2+2gx+2fy+c=0$, we get

$2g=7$, $2f=-9$ and $c=0$

i.e. $g=\frac{7}{2}$, $f=-\frac{9}{2}$ and $c=0$.

We know that centre of circle is given by $(-g,-f)$ and radius r is given by $\sqrt{g^2+f^2-c}$ Therefore, centre of circle (8) is $\left(-\frac{7}{2},\frac{9}{2}\right)$ and radius r of circle (8) is

$$r=\sqrt{\left(\frac{7}{2}\right)^2+\left(-\frac{9}{2}\right)^2-0}$$

$$\Rightarrow \quad r=\sqrt{\frac{49}{4}+\frac{81}{4}}$$

$$\Rightarrow \quad r=\sqrt{\frac{49+81}{4}}=\sqrt{\frac{130}{4}}$$

$$\Rightarrow \quad r=\sqrt{\frac{65}{2}}$$

Example 56. Find the equations of circles if their centres and radii are as follow:

(i) $(0,0)$, 2 (ii) $(2,0)$, 5 (iii) $(0,-3)$, 3
(iv) $(8,-4)$, 1 (v) $(3,6)$, 6 (vi) $(-2,-5)$, 10

Sol.

(i) Given that centre of circle is $(0,0)$ and radius is 2 i.e. $h=0$, $k=0$ and $r=2$.
We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$$\Rightarrow \quad (x-0)^2+(y-0)^2=2^2$$

$\Rightarrow \quad x^2+y^2=4$

$\Rightarrow \quad x^2+y^2-4=0$

which is the required equation of circle.

(ii) Given that centre of circle is $(2,0)$ and radius is 5 i.e. $h=2$, $k=0$ and $r=5$.

We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$\Rightarrow \quad (x-2)^2+(y-0)^2=5^2$

$\Rightarrow \quad x^2+4-4x+y^2=25$

$\Rightarrow \quad x^2+y^2-4x-21=0$

which is the required equation of circle.

(iii) Given that centre of circle is $(0,-3)$ and radius is 3 i.e. $h=0$, $k=-3$ and $r=3$.

We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$\Rightarrow \quad (x-0)^2+(y-(-3))^2=3^2$

$\Rightarrow \quad x^2+(y+3)^2=3^2$

$\Rightarrow \quad x^2+y^2+9+6y=9$

$\Rightarrow \quad x^2+y^2+6y=0$

which is the required equation of circle.

(iv) Given that centre of circle is $(8,-4)$ and radius is 1 i.e. $h=8$, $k=-4$ and $r=1$.

We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$\Rightarrow \quad (x-8)^2+(y-(-4))^2=1^2$

$\Rightarrow \quad (x-8)^2+(y+4)^2=1^2$

$\Rightarrow \quad x^2+64-16x+y^2+16+8y=1$

$\Rightarrow \quad x^2+y^2-16x+8y+79=0$

which is the required equation of circle.

(v) Given that centre of circle is $(3,6)$ and radius is 6 i.e. $h=3$, $k=6$ and $r=6$.

We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$\Rightarrow \quad (x-3)^2+(y-6)^2=6^2$

$\Rightarrow \quad x^2+9-6x+y^2+36-12y=36$

$\Rightarrow \quad x^2+y^2-6x-12y+9=0$

which is the required equation of circle.

(vi) Given that centre of circle is $(-2,-5)$ and radius is 10 i.e. $h=-2$, $k=-5$ and $r=10$.

We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$\Rightarrow \quad (x-(-2))^2+(y-(-5))^2=10^2$

$\Rightarrow \quad (x+2)^2+(y+5)^2=100$

$\Rightarrow \quad x^2+4+4x+y^2+25+10y=100$

$\Rightarrow \quad x^2+y^2+4x+10y-71=0$

which is the required equation of circle.

Example 57. Find the equation of circle whose centre coincides with origin and radius is 4.

Sol. Given that centre of circle coincides with origin i.e. $h=0$, $k=0$ and radius $r=4$.

We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$$\Rightarrow \quad (x-0)^2+(y-0)^2=4^2$$

$$\Rightarrow \quad x^2+y^2=16$$

$$\Rightarrow \quad x^2+y^2-16=0$$

which is the required equation of circle.

Example 58. Find the equation of circle whose centre is $(-5,4)$ and passes through the origin.

Sol. Given that centre of circle is $(-5,4)$ i.e. $h=-5$, $k=4$.

Also the circle passes through origin.

Therefore radius is given by

$$r=\sqrt{(-5-0)^2+(4-0)^2}$$

$$\Rightarrow \quad r=\sqrt{25+16}=\sqrt{41}$$

We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$$\Rightarrow \quad (x+5)^2+(y-4)^2=\left(\sqrt{41}\right)^2$$

$$\Rightarrow \quad x^2+25+10x+y^2+16-8y=41$$

$$\Rightarrow \quad x^2+y^2+10x-8y=0$$

which is the required equation of circle.

Example 59. Find the equation of circle with radius 4 whose centre lies on X-axis and passes through the point $(2,-4)$.

Sol. Given that centre of circle lies on X-axis. Let the centre is $(h,0)$ i.e. $k=0$. Also, given that radius $r=4$. We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$$\Rightarrow \quad (x-h)^2+(y-0)^2=(4)^2$$

$$\Rightarrow \quad x^2+h^2-2hx+y^2=16$$

$$\Rightarrow \quad x^2+y^2-2hx+h^2-16=0 \qquad \ldots\ldots \qquad (1)$$

Also the circle passes through $(2,-4)$. Put $x=2$ and $y=-4$ in (1), we get

$$2^2+(-4)^2-2h(2)+h^2-16=0$$

$$\Rightarrow \quad 4+16-4h+h^2-16=0$$

$$\Rightarrow \quad h^2-4h+4=0$$

$$\Rightarrow \quad h^2-2h-2h+4=0$$

$$\Rightarrow \quad h(h-2)-2(h-2)=0$$

$$\Rightarrow \quad (h-2)(h-2)=0$$

$$\Rightarrow \quad h-2=0$$

$$\Rightarrow \quad h=2$$

Put this value in (1), we get

$$x^2+y^2-2(2)x+2^2-16=0$$

$\Rightarrow \quad x^2+y^2-4x+4-16=0$

$\Rightarrow \quad x^2+y^2-4x-12=0$

which is the required equation of circle.

Example 60. Find the equation of circle with radius 3 whose centre lies on Y-axis and passes through the point $(-3,1)$.

Sol. Given that centre of circle lies on Y-axis. Let the centre is $(0,k)$ i.e. $h=0$.

Also, given that radius $r=3$ We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$\Rightarrow \quad (x-0)^2+(y-k)^2=(3)^2$

$\Rightarrow \quad x^2+y^2+k^2-2ky=9$

$\Rightarrow \quad x^2+y^2+k^2-2ky-9=0 \qquad \ldots.\ (1)$

Also the circle passes through $(-3,1)$. Put $x=-3$ and $y=1$ in (1), we get

$$(-3)^2+(1)^2+k^2-2k(1)-9=0$$

$\Rightarrow \quad 9+1+k^2-2k-9=0$

$\Rightarrow \quad k^2-2k+1=0$

$\Rightarrow \quad k^2-k-k+1=0$

$\Rightarrow \quad k(k-1)-1(k-1)=0$

$\Rightarrow \quad (k-1)(k-1)=0$

$\Rightarrow \quad k-1=0$

$\Rightarrow \quad k=1$

Put this value in (1), we get

$$x^2+y^2+1^2-2(1)y-9=0$$

$\Rightarrow \quad x^2+y^2+1-2y-9=0$

$\Rightarrow \quad x^2+y^2-2y-8=0$

which is the required equation of circle.

Example 61. Find the equation of circle which touches the Y-axis with centre $(-3,1)$.

Sol. Given that centre of circle is $(-3,1)$ i.e. $h=-3$ and $y=1$.

Also the circle touches the Y-axis. Therefore $r=|h|=|-3|=3$

We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$\Rightarrow \quad (x-(-3))^2+(y-1)^2=(3)^2$

$\Rightarrow \quad (x+3)^2+(y-1)^2=9$

$\Rightarrow \quad x^2+9+6x+y^2+1-2y-9=0$

$\Rightarrow \quad x^2+y^2+6x-2y+1=0$

which is the required equation of circle.

Example 62. Find the equation of circles if end points of their diameters are as follow:

(i) $(1,5)$ and $(3,6)$ (ii) $(1,0)$ and $(-2,-5)$

(iii) $(0,0)$ and $(8,-6)$ (iv) $(-3,2)$ and $(-7,9)$

Sol.

(i) Given that end points of diameter of circle are $(1,5)$ and $(3,6)$.

Comparing these points with (x_1,y_1) and (x_2,y_2) respectively, we get $x_1=1$, $y_1=5$, $x_2=3$ and $y_2=6$. We know that the equation of circle in diametric form is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

$\Rightarrow \quad (x-1)(x-3)+(y-5)(y-6)=0$

$\Rightarrow \quad x^2-3x-x+3+y^2-6y-5y+30=0$

$\Rightarrow \quad x^2+y^2-4x-11y+33=0$

which is the required equation of circle.

(ii) Given that end points of diameter of circle are $(1,0)$ and $(-2,-5)$.

Comparing these points with (x_1,y_1) and (x_2,y_2) respectively, we get $x_1=1$, $y_1=0$, $x_2=-2$ and $y_2=-5$.

We know that the equation of circle in diametric form is

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

$\Rightarrow \quad (x-1)(x-(-2))+(y-0)(y-(-5))=0$

$\Rightarrow \quad (x-1)(x+2)+y(y+5)=0$

$\Rightarrow \quad x^2+2x-x-2+y^2+5y=0$

$\Rightarrow \quad x^2+y^2+x+5y-2=0$

which is the required equation of circle.

(iii) Given that end points of diameter of circle are $(0,0)$ and $(8,-6)$.

Comparing these points with (x_1,y_1) and (x_2,y_2) respectively, we get $x_1=0$, $y_1=0$, $x_2=8$ and $y_2=-6$. We know that the equation of circle in diametric form is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

$\Rightarrow \quad (x-0)(x-8)+(y-0)(y-(-6))=0$

$\Rightarrow \quad x(x-8)+y(y+6)=0$

$\Rightarrow \quad x^2-8x+y^2+6y=0$

$\Rightarrow \quad x^2+y^2-8x+6y=0$

which is the required equation of circle.

(iv) Given that end points of diameter of circle are $(-3,2)$ and $(-7,9)$.

Comparing these points with (x_1,y_1) and (x_2,y_2) respectively, we get $x_1=-3$, $y_1=2$, $x_2=-7$ and $y_2=9$.

We know that the equation of circle in diametric form is

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

$\Rightarrow \quad (x-(-3))(x-(-7))+(y-2)(y-9)=0$

$\Rightarrow \quad (x+3)(x+7)+(y-2)(y-9)=0$

$\Rightarrow \quad x^2+7x+3x+21+y^2-9y-2y+18=0$

$\Rightarrow \quad x^2+y^2+10x-11y+39=0$

which is the required equation of circle.

EXERCISE - III

1. Equation of circle with centre at (2,0) and radius 7 is:

(a) $x^2 + 4 - 4x + y^2 = 14$ (b) $x^2 + 4 - 4x + y^2 = 49$

(c) $x^2 - 4 + 4x + y^2 = 49$ (d) None of these

2. Equation of a circle whose centre is origin and radius v is:

(a) $x^2 + y^2 + 2gn + 2fy + c = 0$ (b) $x^2 + y^2 = v^2$

(c) $x^2 - 2vx + y^2 = v^2$ (d) $x^2 + y^2 = 0$

3. Equation of circle in diametric form is:

(a) $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ (b) $(x - x_1)(y - y_1) + (x - x_2)(y - y_2) = 0$

(c) $(x - y_1)(x - x_2) + (y - x_1)(y - x_2) = 0$ (d) $(x - y)(x - y) + (y_1 - x_1)(y_2 - x_2) = 0$

4. Find the centre and radius of the following:

(i) $9x^2 + 9y^2 - 12x - 30y + 24 = 0$

(ii) $x^2 + y^2 - 6y - 24 = 0$

(iii) $x^2 + y^2 + 20x - 5 = 0$

5. Find the equation of circles of their centre and radii are as follows:

(i) (8, 8) , 2 (ii) (6, 3) , 6

6. Find the equation of the circle with radius 3 whose centre lies on Y-axis and passes through the point (2, -2).

7. Find the equations of circles if end points of their diameters are as follows:

(i) (2, 6) and (3, 16) (ii) (-1, -1) and (4, 5)

ANSWERS

1. (b) 2. (b) 3.(a)

4. (i) $(2/3, 5/3); \sqrt{5/9}$ (ii) $(0,3); \sqrt{33}$ (iii) $(-10,0); \sqrt{105}$

5. (i) $x^2 + y^2 - 16x - 16y + 124 = 0$ (ii) $x^2 + y^2 - 12x - 6y + 9 = 0$

6. $x^2 + y^2 - 2(-2+\sqrt{5})y - 4\sqrt{5} = 0;$ $x^2 + y^2 + 2(2+\sqrt{5})y + 4\sqrt{5} = 0;$

7. (i) $x^2 + y^2 - 5x - 22y + 102 = 0$ (ii) $x^2 + y^2 - 3x - 4y - 9 = 0$

* * * * * *

UNIT- 4
DIFFERENTIAL CALCULUS

Learning Objectives

- To learn concept of function, limit, and differentiation for function of one variables.
- To learn the differentiation of various functions, their sum and product.
- To learn applications of derivatives in real life applications

4.1 FUNCTIONS

Definition of Function: Let A and B be two non empty sets. A rule $f:A\to B$ (read as f from A to B) is said to be a function if to each element x of A there exists a unique element y of B such that $f(x)=y$.y is called the image of x under the map f. Here x is independent variable and y is dependent variable.

There are mainly two types of functions: Explicit functions and Implicit functions. If y is clearly expressed in the terms of x directly then the function is called Explicit function. e.g. $y=x+20$.

If y can't be expressed in the terms of x directly then the function is called Implicit function. e.g. $a\,x^2+2hxy+by^2=1$.

We may further categorize the functions according to their nature as:

Functions Types	**Algebraic**	**Trigonometric**	**Inverse Trigonometric**	**Exponential**	**Logarithmic**
Examples	$y=x^2+x+1$, $y=x^3-3x+2$, y = $x+1$ etc.	$y=sinx$, $y=cosx$, $y=secx$ etc.	$y=tan^{-1}x$, $y=cos^{-1}x$, $y=cot^{-1}x$ etc.	$y=e^x$, $y=2^x$, $y=5^x$ etc.	$y=log_e x$, $y=log_5 x$, $y=log_{10}x$ etc.

Even Function: A function $f(x)$ is said to be an even function if $f(-x)=f(x)$ for all x.

For Example: x^2 , $x^4+\ 1$,$\cos(x)$ etc.

Odd Function: A function $f(x)$ is said to be an odd function if $f(-x)=-f(x)$ for all x.

For Example: x^3 ,$\sin(x)$,$\tan(x)$ etc.

Periodic Function: A function $f(x)$ is said to be a periodic function if it retains same value after a certain period.

For Example: $sin(x)$, $tan(x)$ etc.

As $\sin(x)=\sin(x+2\pi)=\sin(x+4\pi)=\sin(x+6\pi)$ ...

Therefore $sin(x)$ is a periodic function with period 2π.

Examples to solve functions:

Example 1. If $f(x)\ =\ x^2+1$, find $f(2)$.

Sol. Given that $f(x)\ =\ x^2+1$

Put $x=2$ in function, we get

$f(2)\ =\ 2^2+1=4+1=5$.

Example 2. If $f(x)\ =\ 2\,x^2-4\,x+6$, find $\frac{f(-2)}{f(1)}$.

Sol. Given that $f(x)\ =\ 2\,x^2-4\,x+6$

Put $x=-2$ in function, we get

$f(-2) = 2(-2)^2 - 4(-2) + 6 = 2(4) + 8 + 6 = 8 + 14 = 22$

Again put $x = 1$ in function, we get

$f(1) = 2(1)^2 - 4(1) + 6 = 2 - 4 + 6 = 4$

Therefore $\frac{f(-2)}{f(1)} = \frac{22}{4} = 5.5$

<u>Note</u>:

(i) The symbol "∞" is called infinity.

(ii) $\frac{a}{0}$ is not finite $(where\ a \neq 0)$ and it is represented by ∞ .

(iii) $\frac{a}{\infty} = 0$ if $(a \neq \infty)$.

Indeterminate Forms: The following forms are called indeterminate forms:

$\frac{0}{0}, \frac{\infty}{\infty}, \ \infty^{\infty}, \ 0^0, \ \infty^0, \ \infty - \infty, \ 0 \times \infty \ etc.$

(These forms are meaningless)

<u>Concept of Limits</u>:

A function $f(x)$ is said to have limit l when x tends to a, if for every positive ε (however small) there exists a positive number δ such that $|f(x) - l| < \varepsilon$ for all values of x for which $0 < |x - a| < \delta$ and it is represented as

$\lim_{x \to a} f(x) = l$

Some basic properties on Limits:

(i) $\lim_{x\to a} K = K$ where K is some constant.

(ii) $\lim_{x\to a} K.f(x) = K.\lim_{x\to a} f(x)$ where K is some constant.

(iii) $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(iv) $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

(v) $\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$

(vi) $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided that $\lim_{x\to a} g(x) \neq 0$

(vii) $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$

Methods of finding the limits of the functions:

1) Direct Substitution Method
2) Factorization Method
3) Rationalization Method etc.

Some Standard Limits Formulas:

1) $\lim_{x\to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$

2) $\lim_{x\to \infty} \left(1 + \frac{1}{x}\right)^x = e$

3) $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

4) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$

5) $\lim_{x\to 0}\frac{e^x-1}{x}=\log_e e=1$

6) $\lim_{x\to 0}\sin x=0$

7) $\lim_{x\to 0}\tan x=0$

8) $\lim_{x\to 0}\cos x=1$

9) $\lim_{x\to 0}\frac{\sin x}{x}=1$

10) $\lim_{x\to 0}\frac{\tan x}{x}=1$

Some Solved Examples on limits:

Example 3. Evaluate $\lim_{x\to -1}\left(1+x+x^2+x^3\right)$.

Sol. $\lim_{x\to -1}\left(1+x+x^2+x^3\right)=1+(-1)+(-1)^2+(-1)^3=1-1+1-1=0$

Example 4. Evaluate $\lim_{x\to -1}\frac{x^3+6}{x+1}$.

Sol. $\lim_{x\to -1}\frac{x^3+6}{x+1}=\frac{(-1)^3+6}{-1+1}=\frac{-1+6}{0}=\frac{5}{0}=\infty$

Example 5. Evaluate $\lim_{x\to 2}\frac{x^3-8}{x-2}$.

Sol. $\lim_{x\to 2}\frac{x^3-8}{x-2}=\frac{2^3-8}{2-2}=\frac{8-8}{2-2}=\frac{0}{0}$ $\left(\frac{0}{0}\ \textit{form}\right)$

$\therefore\ \lim_{x\to 2}\frac{x^3-8}{x-2}=\lim_{x\to 2}\frac{x^3-2^3}{x-2}$ By factorization method

$$=\lim_{x\to 2}\frac{(x-2)\left(x^2+2^2+2x\right)}{x-2}$$

$$=\lim_{x\to 2}\left(x^2+2^2+2x\right)$$

$$=2^2+2^2+2(2)$$

$$=4+4+4=12$$

Example 6. Evaluate $\lim_{x\to 0}\frac{\sqrt{a+x}-\sqrt{a-x}}{x}$.

Sol. $\lim_{x\to 0}\frac{\sqrt{a+x}-\sqrt{a-x}}{x}=\frac{\sqrt{a+0}-\sqrt{a-0}}{0}=\frac{\sqrt{a}-\sqrt{a}}{0}=\frac{0}{0}$ $\left(\frac{0}{0}\ \textit{form}\right)$

$\therefore\ \lim_{x\to 0}\frac{\sqrt{a+x}-\sqrt{a-x}}{x}$ By rationalization method

$$=\lim_{x\to 0}\frac{\sqrt{a+x}-\sqrt{a-x}}{x}\times\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$$

$$= \lim_{x \to 0} \frac{\left(\sqrt{a+x}\right)^2 - \left(\sqrt{a-x}\right)^2}{x\left(\sqrt{a+x}+\sqrt{a-x}\right)}$$

$$= \lim_{x \to 0} \frac{a+x-a+x}{x\left(\sqrt{a+x}+\sqrt{a-x}\right)}$$

$$= \lim_{x \to 0} \frac{2x}{x\left(\sqrt{a+x}+\sqrt{a-x}\right)}$$

$$= \lim_{x \to 0} \frac{2}{\left(\sqrt{a+x}+\sqrt{a-x}\right)}$$

$$= \frac{2}{\left(\sqrt{a+0}+\sqrt{a-0}\right)}$$

$$= \frac{2}{2\sqrt{a}} = \frac{1}{\sqrt{a}}$$

Method of evaluation of algebraic limits when $x \to \infty$:

Example 7. Evaluate $\lim\limits_{x \to \infty} \dfrac{(x+1)(x+2)}{(x+3)(x+4)}$.

Sol. $$\lim_{x \to \infty} \frac{(x+1)(x+2)}{(x+3)(x+4)} = \frac{(\infty+1)(\infty+2)}{(\infty+3)(\infty+4)} = \frac{\infty}{\infty} \qquad \left(\frac{\infty}{\infty}\ \textit{form}\right)$$

$$\therefore \lim_{x \to \infty} \frac{(x+1)(x+2)}{(x+3)(x+4)} = \lim_{x \to \infty} \frac{x\left(1+\frac{1}{x}\right)x\left(1+\frac{2}{x}\right)}{x\left(1+\frac{3}{x}\right)x\left(1+\frac{4}{x}\right)}$$

$$= \lim_{x \to \infty} \frac{x^2\left(1+\frac{1}{x}\right)\left(1+\frac{2}{x}\right)}{x^2\left(1+\frac{3}{x}\right)\left(1+\frac{4}{x}\right)} = \lim_{x \to \infty} \frac{\left(1+\frac{1}{x}\right)\left(1+\frac{2}{x}\right)}{\left(1+\frac{3}{x}\right)\left(1+\frac{4}{x}\right)}$$

$$= \frac{\left(1+\frac{1}{\infty}\right)\left(1+\frac{2}{\infty}\right)}{\left(1+\frac{3}{\infty}\right)\left(1+\frac{4}{\infty}\right)} = \frac{(1+0)(1+0)}{(1+0)(1+0)} = \frac{1}{1} = 1$$

Example 8. Evaluate $\lim\limits_{x \to \infty} \dfrac{\left(x^2-1\right)}{\left(x^2+2x+1\right)(x+5)}$.

Sol. $$\lim_{x \to \infty} \frac{\left(x^2-1\right)}{\left(x^2+2x+1\right)(x+5)} = \frac{\left(\infty^2-1\right)}{\left(\infty^2+2(\infty)+1\right)(\infty+5)} = \frac{\infty}{\infty} \qquad \left(\frac{\infty}{\infty}\ \textit{form}\right)$$

$$\therefore \lim_{x \to \infty} \frac{\left(x^2-1\right)}{\left(x^2+2x+1\right)(x+5)} = \lim_{x \to \infty} \frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(1+\frac{2}{x}+\frac{1}{x^2}\right)x\left(1+\frac{5}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{\left(1-\frac{1}{x^2}\right)}{x\left(1+\frac{2}{x}+\frac{1}{x^2}\right)\left(1+\frac{5}{x}\right)}$$

$$= \frac{\left(1-\frac{1}{\infty^2}\right)}{\infty\left(1+\frac{2}{\infty}+\frac{1}{\infty^2}\right)\left(1+\frac{5}{\infty}\right)} = \frac{(1-0)}{\infty(1+0+0)(1+0)} = \frac{1}{\infty} = 0$$

Example 9. Evaluate $\lim_{x\to\frac{\pi}{2}} (\sin x - \cos x)$.

Sol. $\lim_{x\to\frac{\pi}{2}} (\sin x - \cos x) = \sin\frac{\pi}{2} - \cos\frac{\pi}{2} = 1 - 0 = 1$

Example 10. Evaluate $\lim_{x\to 0} \frac{\sin 5x}{6x}$.

Sol. $\lim_{x\to 0} \frac{\sin 5x}{6x} = \frac{\sin(5\times 0)}{6\times 0} = \frac{\sin(0)}{0} = \frac{0}{0}$ $\left(\frac{0}{0}\ \textit{form}\right)$

$\therefore\ \lim_{x\to 0} \frac{\sin 5x}{6x} = \lim_{x\to 0} \frac{\sin 5x}{5x} \times \frac{5}{6} = 1 \times \frac{5}{6} = \frac{5}{6}$ $\left[\textit{by}\ \lim_{x\to 0} \frac{\sin x}{x} = 1\right]$

Example 11. Evaluate $\lim_{x\to 0} \frac{9x}{\tan 3x}$.

Sol. $\lim_{x\to 0} \frac{9x}{\tan 3x} = \frac{9\times 0}{\tan(3\times 0)} = \frac{0}{\tan(0)} = \frac{0}{0}$ $\left(\frac{0}{0}\ \textit{form}\right)$

$\therefore\ \lim_{x\to 0} \frac{9x}{\tan 3x} = \lim_{x\to 0} \frac{9x}{\frac{\tan 3x}{3x} \times 3x} = \lim_{x\to 0} \frac{9x}{1\times 3x} = \frac{9}{3} = 3$ $\left[\textit{by}\ \lim_{x\to 0} \frac{\tan x}{x} = 1\right]$

<u>Examples based on trigonometric formulas</u>:

$sinC + sinD = 2sin\frac{C+D}{2}\, cos\frac{C-D}{2}$ & $sinC - sinD = 2cos\frac{C+D}{2}\, sin\frac{C-D}{2}$

Example 12. Evaluate $\lim_{x\to 0} \frac{\sin 4x - \sin 2x}{\sin 7x + \sin 3x}$.

Sol. $\lim_{x\to 0} \frac{\sin 4x - \sin 2x}{\sin 7x + \sin 3x} = \frac{\sin 0 - \sin 0}{\sin 0 + \sin 0} = \frac{0-0}{0+0} = \frac{0}{0}$ $\left(\frac{0}{0}\ \textit{form}\right)$

$$\therefore\ \lim_{x\to 0} \frac{\sin 4x - \sin 2x}{\sin 7x + \sin 3x} = \lim_{x\to 0} \frac{2\cos\left(\frac{4x+2x}{2}\right)\sin\left(\frac{4x-2x}{2}\right)}{2\sin\left(\frac{7x+3x}{2}\right)\cos\left(\frac{7x-3x}{2}\right)}$$

$$= \lim_{x\to 0} \frac{\cos 3x\ \sin x}{\sin 5x\ \cos 2x} = \lim_{x\to 0} \frac{\cos 3x \times \frac{\sin x}{x} \times x}{\frac{\sin 5x}{5x} \times 5x \times \cos 2x}$$

$$= \lim_{x\to 0} \frac{\cos 0 \times 1 \times x}{1\times 5x \times \cos 0} = \lim_{x\to 0} \frac{1\times 1}{1\times 5\times 1} = \frac{1}{5}$$

Examples based on $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\log_e a$ **:**

Example 13. Evaluate $\lim\limits_{x\to 0}\dfrac{2^x-1}{x}$.

Sol. Applying the above formula, we get

$$\lim_{x\to 0}\frac{2^x-1}{x}=\log_e 2 \qquad (here\ a=2)$$

Example 14. Evaluate $\lim\limits_{x\to 0}\dfrac{4^x-3^x}{\sin x}$.

Sol. $\therefore \lim\limits_{x\to 0}\dfrac{4^x-3^x}{\sin x}=\lim\limits_{x\to 0}\dfrac{4^x-1+1-3^x}{\dfrac{\sin x}{x}\times x}=\lim\limits_{x\to 0}\dfrac{(4^x-1)-(3^x-1)}{1\times x}$

$$=\lim_{x\to 0}\left[\left(\frac{4^x-1}{x}\right)-\left(\frac{3^x-1}{x}\right)\right]$$

$$=\log_e 4-\log_e 3=\log_e\frac{4}{3} \qquad \left(\because \log_e a-\log_e b=\log_e\frac{a}{b}\right)$$

Example 15. Evaluate $\lim\limits_{x\to 0}\dfrac{e^{x^2}-1}{x\tan x}$.

Sol. $\lim\limits_{x\to 0}\dfrac{e^{x^2}-1}{x\tan x}=\lim\limits_{x\to 0}\left[\dfrac{e^{x^2}-1}{x^2}\times\dfrac{x^2}{x\tan x}\right]$

$$=\log_e e\times\lim_{x\to 0}\frac{x^2}{x\tan x}=1\times\lim_{x\to 0}\frac{x}{\tan x}=1 \qquad (\because \log_e e=1)$$

EXERCISE-I

1. If $f(x) = x^3+2x^2-3x+1$, find $f(-1)$.
2. If $f(x) = x^2+x+1$, find $f(2).f(3)$.
3. The limit of $f(x)=\dfrac{cosx}{x}$ as $x\to 0$ is ....

   a. 1 b. 0 c. -1 d. Does not exist
4. Evaluate $\lim\limits_{x\to 2}\dfrac{x^3-8}{x+2}$.
5. Evaluate $\lim\limits_{x\to 5}\dfrac{x^2-25}{x-5}$.
6. Evaluate $\lim\limits_{x\to 3}\dfrac{x-3}{x^2-9}$.
7. Evaluate $\lim\limits_{x\to 1}\dfrac{x^2+x-2}{x^2-4x+3}$.
8. Evaluate $\lim\limits_{x\to 3}\dfrac{\sqrt{x}-\sqrt{3}}{x-3}$.
9. Evaluate $\lim\limits_{x\to 0}\dfrac{\sqrt{2+x}-\sqrt{2-x}}{2x}$.
10. Evaluate $\lim\limits_{x\to 4}\dfrac{x^3-64}{x-4}$.

11. Evaluate $\lim_{x \to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2}$.

12. Evaluate $\lim_{x \to 5} \frac{x^{\frac{1}{3}}-5^{\frac{1}{3}}}{x-5}$.

13. Evaluate $\lim_{x \to \infty} \frac{x(x-5)}{x^2+5}$.

14. Evaluate $\lim_{x \to \infty} \frac{(2x+5)(x+3)}{(3x^2+2x-1)(x+9)}$.

15. Evaluate $\lim_{x \to \infty} \frac{(x^5+x^3+5)}{(x^2-9)(x+8)}$.

16. Evaluate $\lim_{x \to \infty} \frac{(x^7+2x^2-6)}{(x^3+5)(x^2+x+1)}$.

17. Evaluate $\lim_{x \to 0} \frac{4x}{\sin 2x}$.

18. Evaluate $\lim_{x \to 0} \frac{\tan 6x}{3x}$.

19. Evaluate $\lim_{x \to 0} \frac{\tan 4x}{\sin 2x}$.

20. Evaluate $\lim_{x \to 0} \frac{\sin 3x-\tan 2x}{x}$.

21. Evaluate $\lim_{x \to 0} \frac{\tan 4x+2x}{2x}$.

22. Evaluate $\lim_{x \to 0} \frac{\sin 6x+\sin 4x}{\sin 5x-\sin 3x}$.

23. Evaluate $\lim_{x \to 0} \frac{\sin 4x-x}{6x-\sin 3x}$.

24. Evaluate $\lim_{x \to 0} \frac{5^x-1}{x}$.

25. Evaluate $\lim_{x \to 0} \frac{3^x-2^x}{x}$.

26. Evaluate $\lim_{x \to 0} \frac{5^x-3^x}{\tan x}$.

27. Evaluate $\lim_{x \to 0} \frac{a^x-1}{b^x-1}$.

28. Evaluate $\lim_{x \to 0} \frac{2^x-1}{3^x-1}$.

29. Evaluate $\lim_{x \to 0} \frac{a^{\tan x}-1}{\tan x}$.

30. Evaluate $\lim_{x \to 0} \frac{e^{\sin x}-1}{x}$.

ANSWERS

1. 5	2. 91	3. d	4. 0

5. 10	6. $\frac{1}{6}$	7. $-\frac{3}{2}$	8. $\frac{1}{2\sqrt{3}}$
9. $\frac{1}{2\sqrt{2}}$	10. 48	11. $\frac{1}{2\sqrt{2}}$	12. $\frac{1}{3\,(5)^{2/3}}$
13. 1	14. 0	15. ∞	16. ∞
17. 2	18. 2	19. 2	20. 1
21. 3	22. 5	23. 1	24. $log_e 5$
25. $\log\frac{3}{2}$	26. $\log\frac{5}{3}$	27. $\frac{\log a}{\log b}$	28. $\frac{\log 2}{\log 3}$
29. $log_e a$	30. 1		

4.2 DIFFERENTIATION

Increment: Increment is the quantity by which the value of a variable changes. It may be positive or negative. e.g. suppose the value of a variable x changes from 5 to 5.3 then 0.3 is the increment in x. Similarly, if the value of variable x changes from 5 to 4.5 then -0.5 is the increment in x.

Usually δx represents the increment in x, δy represents the increment in y, δz represents the increment in z etc.

Derivative or Differential Co-efficient: If y is a function of x. Let δx be the increment in x and δy be the corresponding increment in y, then $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ $(\textit{if it exists})$ is called the derivative or differential co-efficient of y with respect to x and is dented by $\frac{dy}{dx}$.

i.e. $$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

Differentiation:

Let $y = f(x)$ (1)

Let δx be the increment in x and δy be the corresponding increment in y, then

$$y + \delta y = f(x + \delta x) \qquad (2)$$

Subtracting equation (1) from equation (2), we get

$$y + \delta y - y = f(x + \delta x) - f(x)$$

$$\Rightarrow \quad \delta y = f(x + \delta x) - f(x)$$

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Taking limit $\delta x \to 0$ on both sides, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

If this limit exists, we write it as

$$\frac{dy}{dx} = f'(x)$$

where $f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$.

This is called the differentiation or derivative of the function $f(x)$ with respect to x.

Notations: The first order derivative of the function $f(x)$ with respect to x can be represented in the following ways:

$$\frac{d}{dx}(f(x)),\ \frac{df}{dx},\ f'(x),\ f_1(x)\ etc.$$

Similarly, the first order derivative of y with respect to x can be represented as:

$$\frac{dy}{dx},\ y',\ y_1\ etc.$$

Physical Interpretation of Derivatives:

Let the variable t represents the time and the function $f(t)$ represents the distance travelled in time t.

We know that $Speed = \frac{Distance\ travelled}{Time\ taken}$

If time interval is between $'a'$ & $'a+h'$. Here h be increment in a. Then the speed in that interval is given by

$$\frac{Distance\ travelled\ upto\ time\ (a+h) - Distance\ travelled\ upto\ time\ (a)}{length\ of\ time\ interval}$$

$$\frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$$

If we take $h \to 0$ then $\frac{f(a+h)-f(a)}{h}$ approaches the speed at time $t=a$. Thus we can say that derivative is related in the similar way as speed is related to the distance travelled by a moving particle.

Some Properties of Differentiation:

If $f(x)$ and $g(x)$ are differentiable functions, then

(i) $\frac{d}{dx}(K)=0$ where K is some constant.

(ii) $\frac{d}{dx}(K.f(x))=K.\frac{d}{dx}(f(x))$ where K is some constant.

(iii) $\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}(f(x))+\frac{d}{dx}(g(x))$

(iv) $\frac{d}{dx}[f(x)-g(x)]=\frac{d}{dx}(f(x))-\frac{d}{dx}(g(x))$

(v) $\frac{d}{dx}[f(x).g(x)]=f(x).\frac{d}{dx}(g(x))+g(x).\frac{d}{dx}(f(x))$

This property is known as Product Rule of differentiation.

(vi) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}(f(x))-f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$ provided that $g(x) \neq 0$

This property is known as Quotient Rule of differentiation.

Differentiation of standard functions:

(i) $\frac{d}{dx}\left(x^n\right)=n\,x^{n-1}$. This is known as power formula, here n is any real number.

(ii) $\frac{d}{dx}(\sin x)=\cos x$

(iii) $\frac{d}{dx}(\cos x)=-\sin x$

(iv) $\frac{d}{dx}(\tan x)=\sec^2 x$

(v) $\frac{d}{dx}(\sec x)=\sec x\tan x$

(vi) $\frac{d}{dx}(\cos ecx)=-\cos ecx\cot x$

(vii) $\frac{d}{dx}\left(a^x\right)=a^x\log_e a \qquad here\ a>0\ \&\ a\neq 1$

(viii) $\frac{d}{dx}\left(e^x\right)=e^x\log_e e=e^x$

(ix) $\frac{d}{dx}(\log_e x)=\frac{1}{x}$

(x) $\frac{d}{dx}(\log_a x)=\frac{1}{x}\log_a e$

(xi) $\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}$

(xii) $\frac{d}{dx}\left(\cos^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}}$

(xiii) $\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$

(xiv) $\frac{d}{dx}\left(\cot^{-1}x\right)=\frac{-1}{1+x^2}$

(xv) $\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{|x|\sqrt{x^2-1}}$

(xvi) $\frac{d}{dx}\left(\cos ec^{-1}x\right)=\frac{-1}{|x|\sqrt{x^2-1}}$

Example 16. Differentiate $y=x^{10}$ with respect to x.

Sol. Given that $y=x^{10}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{10}\right)=10\,x^9$$

4.3 DIFFERENTIATION OF SUM, PRODUCT AND QUOTIENT OF FUNCTIONS

Differentiation of sum of two or more functions, product of two or more functions and quotient of two or more functions are explained with following examples as:

Example 17. Differentiate $y=5-x^6$ with respect to x.

Sol. Given that $y=5-x^6$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(5-x^{6}\right)=\frac{d}{dx}(5)-\frac{d}{dx}\left(x^{6}\right)$$

$$=0-6x^{5}=-6x^{5}$$

Example 18. Differentiate $y=\sqrt{x}-\frac{1}{\sqrt{x}}$ with respect to x.

Sol. Given that $y=\sqrt{x}-\frac{1}{\sqrt{x}}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)=\frac{d}{dx}\left(\sqrt{x}\right)-\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right)$$

$$=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)-\frac{d}{dx}\left(x^{-\frac{1}{2}}\right)=\frac{1}{2}x^{\frac{1}{2}-1}-\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$$

$$=\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}x^{-\frac{3}{2}}$$

Example 19. Differentiate $y=\sin x-e^{x}+2^{x}$ with respect to x.

Sol. Given that $y=\sin x-e^{x}+2^{x}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sin x-e^{x}+2^{x}\right)=\frac{d}{dx}(\sin x)-\frac{d}{dx}\left(e^{x}\right)+\frac{d}{dx}\left(2^{x}\right)$$

$$=\cos x-e^{x}+2^{x}\log_{e}2$$

Example 20. Differentiate $y=e^{x}.a^{x}+2x^{3}-\log x$ with respect to x.

Sol. Given that $y=e^{x}.a^{x}+2x^{3}-\log x=(ea)^{x}+2x^{3}-\log x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left((ea)^{x}+2x^{3}-\log x\right)=\frac{d}{dx}(ea)^{x}+2\frac{d}{dx}\left(x^{3}\right)-\frac{d}{dx}(\log x)$$

$$=(ea)^{x}\log_{e}(ea)+2\left(3x^{2}\right)-\frac{1}{x}=(ea)^{x}\log_{e}(ea)+6x^{2}-\frac{1}{x}$$

<u>**Chain Rule**</u>: If $f(x)$ and $g(x)$ are two differentiable functions then

$$\frac{d}{dx}\left(f(g(x))\right)=f'(g(x)).\frac{d}{dx}(g(x))=f'(g(x)).g'(x)$$

So, we may generalize our basic formulas as:

(i) $\frac{d}{dx}\left((f(x))^{n}\right)=n\left(f(x)\right)^{n-1}.f'(x)$ here n is any real number.

(ii) $\frac{d}{dx}\left(\sin(f(x))\right)=\cos(f(x)).f'(x)$ etc.

<u>**Examples based on Chain Rule**</u>**:**

Example 21. Differentiate $y=\sin(2x+1)$ with respect to x.

Sol. Given that $y=\sin(2x+1)$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\sin(2x+1)=\cos(2x+1).\frac{d}{dx}(2x+1)$$

$$=\cos(2x+1).(2\times1+0)=2\cos(2x+1)$$

Example 22. Differentiate $y=\tan(\cos x)$ with respect to x.

Sol. Given that $y=\tan(\cos x)$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\tan(\cos x)=\sec^2(\cos x).\frac{d}{dx}(\cos x)$$

$$=\sec^2(\cos x).(-\sin x)=-\sin x.\sec^2(\cos x)$$

Example 23. Differentiate $r=\sin(\log(5s))$ with respect to s.

Sol. Given that $r=\sin(\log(5s))$

Differentiating it with respect to s, we get

$$\frac{dr}{ds}=\frac{d}{ds}(\sin(\log(5s)))=\cos(\log(5s)).\frac{d}{ds}(\log(5s))$$

$$=\cos(\log(5s)).\frac{1}{5s}.\frac{d}{ds}(5s)=\cos(\log(5s)).\frac{1}{5s}\times 5=\frac{\cos(\log(5s))}{5}$$

<u>Examples based on Product Rule</u>:

Example 24. Differentiate $y=x\cos x$ with respect to x.

Sol. Given that $y=x\cos x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(x\cos x)=x.\frac{d}{dx}(\cos x)+\cos x\frac{dx}{dx}$$

$$=x.(-\sin x)+\cos x.1=-x\sin x+\cos x$$

Example 25. Differentiate $y=\log x\tan x$ with respect to x.

Sol. Given that $y=\log x\tan x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(\log x\tan x)=\log x.\frac{d}{dx}(\tan x)+\tan x\frac{d}{dx}(\log x)$$

$$=\log x\times(\sec^2 x)+\tan x\times\frac{1}{x}=\log x.\sec^2 x+\frac{\tan x}{x}$$

<u>Examples based on Quotient Rule</u>:

Example 26. Differentiate $y=\frac{\sin x}{x}$ with respect to x.

Sol. Given that $y=\frac{\sin x}{x}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\sin x}{x}\right)=\frac{x.\frac{d}{dx}(\sin x)-\sin x.\frac{dx}{dx}}{x^2}$$

$$=\frac{x.\cos x-\sin x.1}{x^2}=\frac{x\cos x-\sin x}{x^2}$$

Example 27. Differentiate $y=\frac{x}{\sqrt{1+x^2}}$ with respect to x.

Sol. Given that $y=\frac{x}{\sqrt{1+x^2}}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x}{\sqrt{1+x^2}}\right)=\frac{\sqrt{1+x^2}\,.\frac{dx}{dx}-x.\frac{d}{dx}\left(\sqrt{1+x^2}\right)}{\left(\sqrt{1+x^2}\right)^2}$$

$$=\frac{\sqrt{1+x^2}\;.1-x.\frac{1}{2}\left(1+x^2\right)^{-\frac{1}{2}}\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)}$$

$$=\frac{\left(\sqrt{1+x^2}-\frac{x}{2\sqrt{1+x^2}}\times 2x\right)}{\left(1+x^2\right)}=\frac{\left(\sqrt{1+x^2}-\frac{x^2}{\sqrt{1+x^2}}\right)}{\left(1+x^2\right)}$$

$$=\frac{\left(\frac{\left(\sqrt{1+x^2}\right)^2-x^2}{\sqrt{1+x^2}}\right)}{\left(1+x^2\right)}$$

$$=\frac{1+x^2-x^2}{\left(1+x^2\right)\sqrt{1+x^2}}=\frac{1}{\left(1+x^2\right)\sqrt{1+x^2}}$$

Examples based on Parametric Form:

Example 28. Evaluate $\frac{dy}{dx}$ if $x=t^2$ *and* $y=2t$.

Sol. Given that $x=t^2$ *and* $y=2t$

Differentiating x with respect to t, we get

$$\frac{dx}{dt}=\frac{d}{dt}\left(t^2\right)=2t$$

Differentiating y with respect to t, we get

$$\frac{dy}{dt}=\frac{d}{dt}(2t)=2$$

$$\therefore\ \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{2}{2t}=\frac{1}{t}\ or\ \frac{1}{\sqrt{x}}$$

Example 29. Evaluate $\frac{dy}{dx}$ if $x=\cos 4\theta$ *and* $y=\sin 2\theta$.

Sol. Given that $x=\cos 4\theta$ *and* $y=\sin 2\theta$

Differentiating x with respect to θ, we get

$$\frac{dx}{d\theta}=\frac{d}{d\theta}(\cos 4\theta)=-\sin 4\theta\times\frac{d}{d\theta}(4\theta)=-4\sin 4\theta$$

Differentiating y with respect to θ, we get

$$\frac{dy}{d\theta}=\frac{d}{d\theta}(\sin 2\theta)=\cos 2\theta\times\frac{d}{d\theta}(2\theta)=2\cos 2\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{2\cos 2\theta}{-4\sin 4\theta} = -\frac{\cos 2\theta}{2\sin 4\theta}$$

Logarithmic Differentiation :

Let $f(x)$ and $g(x)$ are two differentiable function and $y = f(x)^{g(x)}$

To differentiate y, first we take logarithm of y:

$$\log y = \log\left(f(x)^{g(x)}\right)$$

$$\log y = g(x)\log(f(x)) \qquad \left(\log a^b = b\log a\right)$$

Differentiating it with respect to x, we get

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(g(x)\log(f(x)))$$

$$\frac{1}{y}\frac{dy}{dx} = g(x)\frac{d}{dx}(\log f(x)) + \log f(x)\frac{d}{dx}(g(x))$$

$$\frac{1}{y}\frac{dy}{dx} = g(x)\frac{1}{f(x)}f'(x) + \log f(x)g'(x)$$

$$\frac{dy}{dx} = y\left[\frac{g(x)}{f(x)}f'(x) + \log f(x)g'(x)\right]$$

$$\frac{dy}{dx} = f(x)^{g(x)}\left[\frac{g(x)}{f(x)}f'(x) + \log f(x)g'(x)\right]$$

Examples based on Derivative of $f(x)^{g(x)}$ Logarithmic Differentiation :

Example 30. Differentiate $y = x^x$ with respect to x.

Sol. Given that $y = x^x$

Taking logarithm on both sides, we get

$$\log y = \log x^x$$

$$\log y = x\log x \qquad \left(\log a^b = b\log a\right)$$

Differentiating it with respect to x, we get

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(x\log x)$$

$$\frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}(\log x) + \log x\frac{dx}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + \log x.1$$

$$\frac{1}{y}\frac{dy}{dx} = 1 + \log x$$

$$\frac{dy}{dx} = y(1 + \log x)$$

$$\frac{dy}{dx} = x^x(1 + \log x)$$

Example 31. Differentiate $y = x^{\sin x}$ with respect to x.

Sol. Given that $y = x^{\sin x}$

Taking logarithm on both sides, we get

$$\log y = \log x^{\sin x}$$

$$\log y = \sin x . \log x \qquad \left(\log a^b = b\log a\right)$$

Differentiating it with respect to x, we get

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(\sin x . \log x)$$

$$\frac{1}{y}\frac{dy}{dx} = \sin x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\sin x)$$

$$\frac{1}{y}\frac{dy}{dx} = \sin x . \frac{1}{x} + \log x . \cos x$$

$$\frac{dy}{dx} = y\left(\frac{\sin x}{x} + \log x \cos x\right)$$

$$\frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \log x \cos x\right)$$

Example 32. Differentiate $y = (\sin x)^{\cos x}$ with respect to x.

Sol. Given that $y = (\sin x)^{\cos x}$

Taking logarithm on both sides, we get

$$\log y = \log (\sin x)^{\cos x}$$

$$\log y = \cos x \log (\sin x) \qquad \left(\log a^b = b\log a\right)$$

Differentiating it with respect to x, we get

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(\cos x \log (\sin x))$$

$$\frac{1}{y}\frac{dy}{dx} = \cos x \frac{d}{dx}(\log (\sin x)) + \log (\sin x)\frac{d}{dx}(\cos x)$$

$$\frac{1}{y}\frac{dy}{dx} = \cos x \frac{1}{\sin x}\frac{d}{dx}(\sin x) + \log (\sin x).(-\sin x)$$

$$\frac{1}{y}\frac{dy}{dx} = \cot x \cos x - \sin x \log (\sin x)$$

$$\frac{dy}{dx} = y\,(\cot x \cos x - \sin x \log (\sin x))$$

$$\frac{dy}{dx} = (\sin x)^{\cos x}\,(\cot x \cos x - \sin x \log (\sin x))$$

<u>Examples based on Derivative of Infinite Series form</u> :

Example 33. Differentiate $\sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots}}}}$ with respect to x.

Sol. Let $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots}}}}$

$$y = \sqrt{\sin x + y}$$

$$y^2 = \sin x + y$$

Differentiating it with respect to x, we get

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(\sin x + y)$$

$$2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$2y\frac{dy}{dx}-\frac{dy}{dx}=\cos x$$

$$(2y-1)\frac{dy}{dx}=\cos x$$

$$\frac{dy}{dx}=\frac{\cos x}{(2y-1)}$$

Example 34. Differentiate $x^{x^{x^{\cdots}}}$ with respect to x.

Sol. Let $y=x^{x^{x^{\cdots}}}$

$$y=x^y$$

Taking logarithm on both sides, we get

$$\log y=\log x^y$$

$$\log y=y\log x$$

Differentiating it with respect to x, we get

$$\frac{d}{dx}(\log y)=\frac{d}{dx}(y\log x)$$

$$\frac{1}{y}\frac{dy}{dx}=y\frac{d}{dx}(\log x)+\log x\frac{dy}{dx}$$

$$\frac{1}{y}\frac{dy}{dx}-\log x\frac{dy}{dx}=\frac{y}{x}$$

$$\left(\frac{1}{y}-\log x\right)\frac{dy}{dx}=\frac{y}{x}$$

$$\left(\frac{1-y\log x}{y}\right)\frac{dy}{dx}=\frac{y}{x}$$

$$\frac{dy}{dx}=\frac{y^2}{x(1-y\log x)}$$

4.4 SUCCESSIVE DIFFERENTIATION OR HIGHER ORDER DERIVATIVE

Let $y=f(x)$ be a differentiable function, then $\frac{dy}{dx}$ represents the first order derivative of y with respect to x.If we may further differentiate it i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right)$,then it is called second order derivative of y with respect to x. Some other way to represent second order derivative of y with respect to x: $\frac{d^2y}{dx^2}$, y'' , y_2.

So, successive derivatives of y with respect to x can be represented as

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \ldots,\ \frac{d^ny}{dx^n}\ etc.$$

Example 35. If $y=x^8-12x^5+5x^3-12$, find $\frac{d^2y}{dx^2}$.

Sol. Given that $y=x^8-12x^5+5x^3-12$

Differentiating with respect to x , we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^8-12x^5+5x^3-12\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = 8x^7 - 60x^4 + 15x^2$$

Again differentiating with respect to x , we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(8x^7 - 60x^4 + 15x^2\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 56x^6 - 240x^3 + 30x$$

Example 36. If $f(x) = x^2.\sin x$, find $f'(0)\, and\, f''\left(\frac{\pi}{2}\right)$.

Sol. Given that $f(x) = x^2.\sin x$

Differentiating with respect to x , we get

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(x^2.\sin x\right)$$

$$\Rightarrow \quad f'(x) = x^2\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}\left(x^2\right)$$

$$\Rightarrow \quad f'(x) = x^2\cos x + \sin x.2x$$

$$\Rightarrow \quad f'(x) = x^2\cos x + 2x\sin x \qquad (1)$$

Again differentiating with respect to x, we get

$$f''(x) = \frac{d}{dx}\left(x^2\cos x + 2x\sin x\right)$$

$$\Rightarrow \quad f''(x) = \frac{d}{dx}\left(x^2\cos x\right) + 2\frac{d}{dx}(x\sin x)$$

$$\Rightarrow \quad f''(x) = x^2\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}\left(x^2\right) + 2\left[x\frac{d}{dx}(\sin x) + \sin x\frac{dx}{dx}\right]$$

$$\Rightarrow \quad f''(x) = -x^2\sin x + 2x\cos x + 2[x\cos x + \sin x]$$

$$\Rightarrow \quad f''(x) = -x^2\sin x + 4x\cos x + 2\sin x \qquad (2)$$

Put $x = 0$ in equation (1), we get

$$f'(0) = (0)^2\cos 0 + 2\times 0\times\sin 0 = 0$$

Put $x = \frac{\pi}{2}$ in equation (2), we get

$$f''\left(\frac{\pi}{2}\right) = -\left(\frac{\pi}{2}\right)^2\sin\left(\frac{\pi}{2}\right) + 4\times\frac{\pi}{2}\times\cos\left(\frac{\pi}{2}\right) + 2\sin\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad f''\left(\frac{\pi}{2}\right) = -\frac{\pi^2}{4} + 2$$

Example 37. If $y = \sin Ax + \cos Ax$, prove that $\frac{d^2y}{dx^2} + A^2\, y = 0$.

Soln. Given that $y = \sin Ax + \cos Ax$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin Ax + \cos Ax)$$

$$\Rightarrow \quad \frac{dy}{dx} = A\cos Ax - A\sin Ax$$

Again differentiating with respect to x , we get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(A\cos Ax - A\sin Ax\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=-A^2\sin Ax - A^2\cos Ax$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=-A^2\left(\sin Ax+\cos Ax\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=-A^2\,y$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}+A^2\,y=0$$

Example 38. If $y=e^{-Ax}$, prove that $\frac{d^2y}{dx^2}+A\frac{dy}{dx}=0$.

Soln. Given that $y=e^{-Ax}$

Differentiating with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{-Ax}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}=e^{-Ax}\frac{d}{dx}\left(-Ax\right)$$

$$\Rightarrow \quad \frac{dy}{dx}=-Ae^{-Ax}$$

$$\Rightarrow \quad \frac{dy}{dx}=-Ay$$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2}=-A\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}+A\frac{dy}{dx}=0$$

EXERCISE-II

1. Differentiate $y=\sqrt{x}$ with respect to x.
2. Differentiate $y=x^{-\frac{5}{2}}$ with respect to x.
3. Differentiate $y=2-x+3x^2$ with respect to x.
4. Differentiate $y=(x+3)(x-1)$ with respect to x.
5. Differentiate $y=2\log x-5\sec x$ with respect to x.
6. Differentiate $y=\frac{x^2+7}{x}$ with respect to x.
7. Differentiate $y=\cos\left(x^2+x+1\right)$ with respect to x.
8. Differentiate $y=\sin^3 x$ with respect to x.
9. Differentiate $y=\cos(\sin x)$ with respect to x.
10. Differentiate $y=\log(\tan x)$ with respect to x.
11. Differentiate $v=e^{5t^2}$ with respect to t.
12. Differentiate $z=2^{s^2+9}$ with respect to s.

13. Differentiate $y= x^2 \sin x$ with respect to x.

14. Differentiate $y= \cos x \log x$ with respect to x.

15. Differentiate $y=\left(3t^2-9\right)2^t$ with respect to t.

16. Differentiate $y=\dfrac{\log(7x)}{x+1}$ with respect to x.

17. Differentiate $y=\dfrac{\log x}{\tan x}$ with respect to x.

18. Differentiate $y=\dfrac{x^2+1}{\sin x}$ with respect to x.

19. Differentiate $y=\dfrac{\tan 2x}{e^{2x}}$ with respect to x.

20. Evaluate $\dfrac{dy}{dx}$ if $x=2t^2+1$ *and* $y=t^3$.

21. Evaluate $\dfrac{dy}{dx}$ if $x=\log 2t$ *and* $y=2\tan t$.

22. Evaluate $\dfrac{dy}{dx}$ if $x=\sec t$ *and* $y=5^t+7t-11$.

23. Differentiate $y= (\cos x)^x$ with respect to x.

24. Differentiate $y= x^{\cos x}$ with respect to x.

25. Differentiate $\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+...}}}}$ with respect to x.

26. Differentiate $\cos x^{\cos x^{\cos x\ ...}}$ with respect to x.

27. If $y=\log(\sin x)+e^{5x}$, find $\dfrac{d^2y}{dx^2}$.

28. If $y=x^3.e^{-2x}$, find $\dfrac{d^2y}{dx^2}$ *at* $x=3$.

29. If $y = sinx + x$, find $\frac{dy}{dx}$ at $x = \frac{\pi}{2}$

    a. $\frac{\pi}{2}$ b. $-\frac{\pi}{2}$ c. 1 d. 0

30. If $y = f(u)$ and $u = x^2 + 1$, then find $\frac{dy}{dx}$

    a. $f'(u).2x$ b. $f(u)x^2$ c. $-f'(u).x^2$ d. $f'(u)$

ANSWERS

1. $\dfrac{1}{2\sqrt{x}}$	2. $-\dfrac{5}{2}x^{-\frac{7}{2}}$	3. $-1+6x$
4. $2x+2$	5. $\dfrac{2}{x}-5\sec x\tan x$	6. $1-7x^{-2}$
7. $-(2x+1)\sin\left(x^2+x+1\right)$	8. $3\sin^2 x.\cos x$	9. $-\sin(\sin x).\cos x$
10. $\dfrac{1}{\sin x\cos x}$	11. $10te^{5t^2}$	12. $2s\ .\ 2^{s^2+9}.\log_e 2$

13. $x^2\cos x+2x\sin x$ 14. $\frac{\cos x}{x}-\log x.\sin x$ 15. $2^t\left\{\left(3t^2-9\right)\log_e 2+6t\right\}$

16. $\frac{(x+1)-x\log(7x)}{x(x+1)^2}$ 17. $\frac{\tan x-x\log x.\sec^2 x}{x\tan^2 x}$ 18. $\frac{2x\sin x-\left(x^2+1\right)\cos x}{\sin^2 x}$

19. $\frac{2\left(\sec^2 2x-\tan 2x\right)}{e^{2x}}$ 20. $\frac{3}{4}t \; or \; \frac{3}{4}\sqrt{\frac{x-1}{2}}$ 21. $2t\sec^2 t$

22. $\frac{5^t\log_e 5+7}{\sec t.\tan t}$ 23. $(\cos x)^x\left(-x\tan x+\log(\cos x)\right)$

24. $x^{\cos x}\left(\frac{\cos x}{x}-\log x\sin x\right)$

25. $\frac{dy}{dx}=\frac{1}{(2y-1)}$ where $y=\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+...}}}}$

26. $\frac{dy}{dx}=\frac{-y^2\tan x}{(1-y\log(\cos x))}$ where $y=\cos x^{\cos x^{\cos x\ ...}}$

27. $-\operatorname{cosec}^2 x+25e^{5x}$ 28. $18e^{-6}$ 29. c 30. a

4.5 APPLICATIONS OF DIFFERENTIAL CALCULUS

(a) Derivative as Rate Measures:

Let y be a function of x, then $\frac{dy}{dx}$ represents the rate of change of y with respect to x.

If $\frac{dy}{dx}>0$ then y increases when x changes and if $\frac{dy}{dx}<0$ then y decreases when x changes.

Some Important Points to Remember:

(i) Usually **t**, **s**, **v** and **a** are used to represent time, displacement, velocity and acceleration respectively.

Also $v=\frac{ds}{dt}$

$$a=\frac{dv}{dt}=\frac{d^2s}{dt^2}=v.\frac{dv}{ds}$$

(ii) If the particle moves in the direction of s increasing, then $v=\frac{ds}{dt}>0$ and if the particle moves in the direction of s decreasing, then $v=\frac{ds}{dt}<0$.

(iii) If $a=0$ then the particle is said to be moving with constant velocity and if $a<0$ then the particle is said to have retardation.

(iv) If $\frac{dy}{dx}=0$ then y is constant.

(v) If $y=f(x)$ be a curve then $\frac{dy}{dx}$ is said to be the slope of the curve. It is also represented by m i.e. $slope=m=\frac{dy}{dx}$.

(vi) If r is the radius, A is the area and C is the circumference of the circle then $A=\pi r^2$ & $C=2\pi r$.

(vii) If r is the radius, S is the surface area and V is the volume of the sphere then $S=4\pi r^2$ & $V=\frac{4}{3}\pi r^3$.

(viii) If r is the radius of base, h is the height, l is the slant length, S is the surface area and V is the volume of the cone then $S=\pi r l+\pi r^2$ & $V=\frac{1}{3}\pi r^2 h$.

(ix) If a is the length of side of a base, S is the surface area and V is the volume of the cube then $S=6a^2$ & $V=a^3$.

Examples Related to Rate Measure:

Example 39. If $y=x^3+5x^2-6x+7$ and x increases at the rate of 3 units per minute, how fast is the slope of the curve changes when $x=2$.

Sol. Let t represents the time.

Given that $y=x^3+5x^2-6x+7$ (1)

and $\frac{dx}{dt}=3$ (2)

Let m be the slope of the curve.

$$\therefore \quad m=\frac{dy}{dx}$$

$$\Rightarrow \quad m=\frac{d}{dx}\left(x^3+5x^2-6x+7\right) \qquad \text{(using (1))}$$

$$\Rightarrow \quad m=3x^2+10x-6$$

Differentiating it with respect to t, we get

$$\frac{dm}{dt}=\frac{d}{dt}\left(3x^2+10x-6\right)$$

$$\Rightarrow \quad \frac{dm}{dt}=(6x+10)\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dm}{dt}=(6x+10).3 \qquad \text{(using (2))}$$

$$\Rightarrow \quad \frac{dm}{dt}=18x+30 \qquad (3)$$

Put $x=2$ in (3), we get

$$\left(\frac{dm}{dt}\right)_{x=2}=18(2)+30=36+30=66$$

Hence the slope of given curve increases at the rate of 66 units per minute when $x=2$.

Example 40. A particle is moving along a straight line such that the displacement s after time t is given by $s=2t^2+t+7$. Find the velocity and acceleration at time $t=20$.

Sol. Let v be the velocity and a be the acceleration of the particle at time t.

Given that the displacement of the particle is $s=2t^2+t+7$

Differentiating it with respect to t, we get

$$\frac{ds}{dt}=\frac{d}{dt}\left(2t^2+t+7\right)$$

$\Rightarrow \qquad v=4t+1 \qquad (1)$

Again differentiating with respect to t, we get

$$\frac{dv}{dt}=\frac{d}{dt}(4t+1)$$

$\Rightarrow \qquad a=4 \qquad (2)$

Put $t=20$ in (1) and (2), we get

$$[v]_{t=20}=4(20)+1=81 \quad \& \quad [a]_{t=20}=4$$

Hence velocity of the particle is 81 and acceleration is 4 when $t=20$.

Example 41. Find the rate of change of the area of the circle with respect to its radius r when $r=4cm.$.

Sol. Given that r be the radius of the circle.

Let A be the area of the circle.

$\therefore \qquad A=\pi r^2$

$$\Rightarrow \qquad \frac{dA}{dr}=\frac{d}{dr}\left(\pi r^2\right)$$

$$\Rightarrow \qquad \frac{dA}{dr}=2\pi r \qquad (1)$$

Put $r=4$ in (1), we get

$$\left(\frac{dA}{dr}\right)_{r=4}=2\pi\times 4=8\pi$$

Hence the rate of change of area of the circle is $8\pi\, cm^2$..

Example 42. Find the rate of change of the surface area of a ball with respect to its radius r.

Sol. Given that r is the radius of the ball.

Let S be the surface area of the ball.

$\therefore \qquad S=4\pi r^2$

$$\Rightarrow \qquad \frac{dS}{dr}=\frac{d}{dr}\left(4\pi r^2\right)$$

$$\Rightarrow \qquad \frac{dS}{dr}=4\pi\times 2r=8\pi r$$

which is the required rate of change of the surface area of a ball with respect to its radius r.

Example 43. The radius of an air bubble increases at the rate of $2cm/\sec$. At what rate is the volume of the bubble increases when the radius is $5cm$?

Sol. Let r be the radius, V be the volume of the bubble and t represents time.

So, by given statement $\frac{dr}{dt}=2cm/\sec$

(1)

and $\qquad V=\frac{4}{3}\pi r^3$

$$\Rightarrow \quad \frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\Rightarrow \quad \frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2 \times \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dV}{dt} = 8\pi r^2 \qquad (2)$$

Put $r = 5$ in (2), we get

$$\left(\frac{dV}{dt}\right)_{r=5} = 8\pi \times (5)^2 = 200\pi$$

Hence, volume of the bubble increases at the rate of $200\pi\, cm^3/\sec$.

Example 44. Find the rate of change of the volume of the cone with respect to the radius of its base.

Sol. Let r be the radius of the base, h be the height and V be the volume of the cone.

$$\therefore \quad V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow \quad \frac{dV}{dr} = \frac{d}{dr}\left(\frac{1}{3}\pi r^2 h\right)$$

$$\Rightarrow \quad \frac{dV}{dr} = \frac{1}{3}\pi h \times 2r = \frac{2}{3}\pi r h.$$

Example 45. Find the rate of change of the surface area of the cone with respect to the radius of its base.

Sol. Let r be the radius of the base, l be the slant length and S be the surface area of the cone.

$$\therefore \quad S = \pi r l + \pi r^2$$

$$\Rightarrow \quad \frac{dS}{dr} = \frac{d}{dr}\left(\pi r l + \pi r^2\right)$$

$$\Rightarrow \quad \frac{dS}{dr} = \pi l + \pi \times 2r$$

$$\Rightarrow \quad \frac{dS}{dr} = \pi l + 2\pi r$$

Example 46. Sand is pouring from a pipe at the rate $10\, cc/\sec$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-fifth of the radius of the base. How fast the height of the sand cone increases when the height is $6\, cm$?

Sol. Let r be the radius of the base, h be the height and V be the volume of the cone at the time t.

So, by given statement $h = \frac{r}{5}$ (1)

and $V = \frac{1}{3}\pi r^2 h$

$$\Rightarrow \quad V = \frac{1}{3}\pi (5h)^2 h = \frac{25}{3}\pi h^3 \qquad \text{(using (1))}$$

$$\Rightarrow \quad \frac{dV}{dt} = \frac{d}{dt}\left(\frac{25}{3}\pi h^3\right) = \frac{25}{3}\pi \times 3h^2 \frac{dh}{dt} = 25\pi h^2 \frac{dh}{dt} \qquad (2)$$

Also, by given statement $\frac{dV}{dt}=10\,cc/\sec$ (3)

From (2) and (3), we get

$$25\pi h^2 \frac{dh}{dt}=10$$

$$\Rightarrow \quad \frac{dh}{dt}=\frac{10}{25\pi h^2}=\frac{2}{5\pi h^2}$$

When $h=6cm$, $\frac{dh}{dt}=\frac{2}{5\pi(6)^2}=\frac{1}{90\pi}$

Hence, the rate of increase of height of the sand cone is $\frac{1}{90\pi}cm/\sec$,when $h=6cm$.

Example 47. The length of edges of a cube increases at the rate of $2cm/\sec$**.** At what rate is the volume of the cube increases when the edge length is $1cm$?

Sol. Let a be the length of edge and V be the volume of the cube at time t.

So, by given statement $\frac{da}{dt}=2\,cm/\sec$

(1)

and $V=a^3$

$$\Rightarrow \quad \frac{dV}{dt}=\frac{d}{dt}\left(a^3\right)$$

$$\Rightarrow \quad \frac{dV}{dt}=3a^2\frac{da}{dt} \qquad \text{(using (1))}$$

$$\Rightarrow \quad \frac{dV}{dt}=6a^2 \qquad (2)$$

Put $a=1$ in (2), we get

$$\left(\frac{dV}{dt}\right)_{a=1}=6(1)^2=6$$

Hence, volume of the cube increases at the rate of $6cm^3/\sec$.

(b) <u>Maxima and Minima</u>

<u>Maximum Value of a Function & Point of Maxima</u>: Let $f(x)$ be a function defined on domain $D\subset R$. Let a be any point of domain D. We say that $f(x)$ has maximum value at a if $f(x)\le f(a)$ for all $x\in D$ and a is called the point of maxima.

e.g. Let $f(x)=-x^2+5$ for all $x\in R$

Now $x^2\ge 0$ for all $x\in R$

$-x^2\le 0$ for all $x\in R$

$-x^2+5\le 5$ for all $x\in R$

i.e. $f(x)\le 5$ for all $x\in R$

Hence 5 is the maximum value of $f(x)$ which is attained at $x=0$. Therefore, $x=0$ is the point of maxima.

<u>Minimum Value of a Function & Point of Minima</u>: Let $f(x)$ be a function defined on domain $D\subset R$. Let a be any point of domain D. We say that $f(x)$ has minimum value at a if $f(x)\ge f(a)$ for all $x\in D$ and a is called the point of minima.

e.g. Let $f(x) = x^2 + 8$ for all $x \in R$

Now $x^2 \geq 0$ for all $x \in R$

$x^2 + 8 \geq 8$ for all $x \in R$

i.e. $f(x) \geq 8$ for all $x \in R$

Hence 8 is the minimum value of $f(x)$ which is attained at $x = 0$. Therefore, $x = 0$ is the point of minima.

Note: We can also attain points of maxima and minima and their corresponding maximum and minimum value of a given function by differential calculus too.

<u>Working Rule to find points of maxima or minima or inflexion by Differential Calculus</u>:

Step No.	**Working Procedure**
1	Put $y = f(x)$
2	Find $\frac{dy}{dx}$
3	Put $\frac{dy}{dx} = 0$ and solve it for x. Let $x_1, x_2, \ldots, x_n$ are the values of x.
4	Find $\frac{d^2y}{dx^2}$.
5	Put the values of x in $\frac{d^2y}{dx^2}$. Suppose $x = x_i$ be any value of x. If $\frac{d^2y}{dx^2} < 0$ at $x = x_i$ then $x = x_i$ is the point of maxima and $f(x_i)$ is the maximum value of $f(x)$. If $\frac{d^2y}{dx^2} > 0$ at $x = x_i$ then $x = x_i$ is the point of minima and $f(x_i)$ is minimum value of $f(x)$. If $\frac{d^2y}{dx^2} = 0$ at $x = x_i$. Find $\frac{d^3y}{dx^3}$. If $\frac{d^3y}{dx^3} \neq 0$ at $x = x_i$ then $x = x_i$ is the point of inflexion.

<u>Examples of Maxima and Minima</u>:

Example 48. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x) = x^3 - 12x^2 + 5$.

Sol. Let $y = f(x) = x^3 - 12x^2 + 5$

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 - 12x^2 + 5\right)$$

$$\frac{dy}{dx} = 3x^2 - 24x$$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(3x^2 - 24x\right)$$

$$\frac{d^2 y}{d x^2} = 6x - 24$$

Put $\frac{dy}{dx} = 0$, we get

$$3x^2 - 24x = 0$$

$$3x(x-8) = 0$$

$$Either \;\; x = 0 \;\; or \;\; x - 8 = 0$$

$$Either \;\; x = 0 \;\; or \;\; x = 8$$

i.e. When $x = 0$:

$$\left(\frac{d^2 y}{d x^2}\right)_{x=0} = (6x - 24)_{x=0} = -24 < 0$$

which shows that $x = 0$ is a point of maxima.

So maximum value of $f(x) = x^3 - 12x^2 + 5$ is

$$(y)_{x=0} = \left(x^3 - 12x^2 + 5\right)_{x=0} = 5$$

When $x = 8$:

$$\left(\frac{d^2 y}{d x^2}\right)_{x=8} = (6x - 24)_{x=8} = 6(8) - 24 = 24 > 0$$

which shows that $x = 8$ is a point of minima.

So minimum value of $f(x) = x^3 - 12x^2 + 5$ is

$$(y)_{x=8} = \left(x^3 - 12x^2 + 5\right)_{x=8} = 8^3 - 12(8)^2 + 5$$

$$= 512 - 768 + 5 = -251$$

Example 49. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x) = \sin x + \cos x \;\; where\, 0 \le x \le \frac{\pi}{2}$.

Sol. Let $y = f(x) = \sin x + \cos x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x + \cos x)$$

$$\frac{dy}{dx} = \cos x - \sin x$$

Again differentiating with respect to x, we get

$$\frac{d^2 y}{d x^2} = \frac{d}{dx}(\cos x - \sin x)$$

$$\frac{d^2 y}{d x^2} = -\sin x - \cos x$$

Put $\frac{dy}{dx} = 0$, we get

$$\cos x - \sin x = 0$$

$$\cos x = \sin x$$

$$\frac{\cos x}{\sin x} = 1$$

$$\cot x = 1$$

$$\cot x = \cot\frac{\pi}{4} \qquad\qquad as\ 0 \le x \le \frac{\pi}{2}$$

$$x = \frac{\pi}{4}$$

When $x = \frac{\pi}{4}$:

$$\left(\frac{d^2y}{dx^2}\right)_{x=\frac{\pi}{4}} = \left(-\sin x - \cos x\right)_{x=\frac{\pi}{4}} = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} < 0$$

which shows that $x = \frac{\pi}{4}$ is a point of maxima.

So maximum value of $f(x) = \sin x + \cos x$ is

$$(y)_{x=\frac{\pi}{4}} = \left(\sin x + \cos x\right)_{x=\frac{\pi}{4}}$$

$$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$$

Example 50. Find two positive numbers x & y such that $x.y = 16$ and the sum $x + y$ is minimum. Also find the minimum value of sum.

Sol. Given that $x.y = 16 \qquad \Rightarrow\ y = \frac{16}{x}$

Let $S = x + y \qquad \Rightarrow\ S = x + \frac{16}{x}$

Differentiating it with respect to x, we get

$$\frac{dS}{dx} = \frac{d}{dx}\left(x + \frac{16}{x}\right)$$

$$\frac{dS}{dx} = 1 + 16\left(-\frac{1}{x^2}\right) = 1 - \frac{16}{x^2}$$

Again differentiating with respect to x, we get

$$\frac{d^2S}{dx^2} = \frac{d}{dx}\left(1 - \frac{16}{x^2}\right)$$

$$\frac{d^2S}{dx^2} = 0 - 16\left(-\frac{2}{x^3}\right) = \frac{32}{x^3}$$

Put $\frac{dS}{dx} = 0$, we get

$$1 - \frac{16}{x^2} = 0$$

$$\frac{x^2 - 16}{x^2} = 0$$

$$x^2 - 16 = 0$$

Either $x = 4$ *or* $x = -4$

$x = -4$ is rejected as x is positive.

When $x = 4$:

$$\left(\frac{d^2 S}{d x^2}\right)_{x=4} = \left(\frac{32}{x^3}\right)_{x=4} = \frac{32}{4^3} = \frac{32}{64} = \frac{1}{2} > 0$$

which shows that $x=4$ is a point of minima.

Now at $x=4$, value of y is :

$$(y)_{x=4} = \left(\frac{16}{x}\right)_{x=4} = \frac{16}{4} = 4$$

Also minimum value of sum $S=x+y$ is

$$(S)_{x=4,\,y=4} = (x+y)_{x=4,\,y=4} = 4+4=8$$

Example 51. Find the dimensions of the rectangle of given area 169 sq**.** cm. whose perimeter is least. Also find its perimeter.

Sol. Let the sides of the rectangle be x and y, A be the area and P be the perimeter.

$\therefore \quad A = x\,y = 169\, sq.c.m. \qquad \Rightarrow y = \frac{169}{x}$

And $\quad P = 2(x+y) \qquad \Rightarrow P = 2\left(x + \frac{169}{x}\right) = 2x + \frac{338}{x}$

Differentiating it with respect to x, we get

$$\frac{dP}{dx} = \frac{d}{dx}\left(2x + \frac{338}{x}\right)$$

$$\frac{dP}{dx} = 2 + 338\left(-\frac{1}{x^2}\right) = 2 - \frac{338}{x^2}$$

Again, differentiating with respect to x, we get

$$\frac{d^2 P}{d x^2} = \frac{d}{dx}\left(2 - \frac{338}{x^2}\right)$$

$$\frac{d^2 P}{d x^2} = 0 - 338\left(-\frac{2}{x^3}\right) = \frac{676}{x^3}$$

Put $\frac{dP}{dx} = 0$, we get

$$2 - \frac{338}{x^2} = 0$$

$$\frac{2x^2 - 338}{x^2} = 0$$

$$2x^2 - 338 = 0$$

$$x^2 = 169$$

Either $x = 13$ *or* $x = -13$

$x = -13$ is rejected as x can’t be negative.

When $x = 13$:

$$\left(\frac{d^2 P}{d x^2}\right)_{x=13} = \left(\frac{676}{x^3}\right)_{x=13} = \frac{676}{(13)^3} = \frac{4}{13} > 0$$

which shows that $x=13$ is a point of minima.

Therefore, Perimeter is least at $x=13$.

Now at $x=13$, value of y is :

$$(y)_{x=13}=\left(\frac{169}{x}\right)_{x=13}=\frac{169}{13}=13$$

Also least value of perimeter $P=2(x+y)$ is

$$(P)_{x=13,\,y=13}=(2x+2y)_{x=13,\,y=13}=26+26=52 \text{ cm}$$

Example 52. Show that among all the rectangles of a given perimeter, the square has the maximum area.

Sol. Let the sides of the rectangle are $x \text{ and } y$, A be the area and P be the given perimeter.

$\therefore \quad P=2(x+y) \quad \Rightarrow P=2x+2y \quad \Rightarrow y=\frac{P-2x}{2}=\frac{P}{2}-x$

and

$$A=xy=\frac{Px}{2}-x^2$$

Differentiating it with respect to x, we get

$$\frac{dA}{dx}=\frac{d}{dx}\left(\frac{Px}{2}-x^2\right)$$

$$\frac{dA}{dx}=\frac{P}{2}-2x$$

Again differentiating with respect to x, we get

$$\frac{d^2A}{dx^2}=\frac{d}{dx}\left(\frac{P}{2}-2x\right)$$

$$\frac{d^2A}{dx^2}=0-2=-2$$

Put $\frac{dA}{dx}=0$, we get

$$\frac{P}{2}-2x=0$$

$$x=\frac{P}{4}$$

When $x=\frac{P}{4}$:

$$\left(\frac{d^2A}{dx^2}\right)_{x=\frac{P}{4}}=-2<0$$

which shows that $x=\frac{P}{4}$ is a point of maxima.

Therefore, Area is maximum at $x=\frac{P}{4}$.

Now at $x=\frac{P}{4}$, value of y is :

$$(y)_{x=\frac{P}{4}}=\left(\frac{P}{2}-x\right)_{x=\frac{P}{4}}=\frac{P}{2}-\frac{P}{4}=\frac{P}{4}$$

$\Rightarrow \quad x=y=\frac{P}{4}$ gives the maximum area.

Hence, among all the rectangles of a given perimeter, the square has the maximum area.

Example 53. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x) = x^3 + 1$.

Sol. Let $y = f(x) = x^3 + 1$

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 + 1\right)$$

$$\frac{dy}{dx} = 3x^2$$

Again, differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(3x^2\right)$$

$$\frac{d^2y}{dx^2} = 6x$$

Put $\frac{dy}{dx} = 0$, we get

$$3x^2 = 0$$

$$\Rightarrow \quad x = 0$$

When $x = 0$:

$$\left(\frac{d^2y}{dx^2}\right)_{x=0} = (6x)_{x=0} = 6 \times 0 = 0$$

To check maxima or minima, we need to find third order derivative of y with respect to x.

So, $$\frac{d^3y}{dx^3} = \frac{d}{dx}(6x)$$

$$\Rightarrow \quad \frac{d^3y}{dx^3} = 6 \neq 0$$

which shows that $x = 0$ is neither a point of maxima nor a point of minima, hence the given function has neither maximum value nor minimum value.

EXERCISE-III

1. If $y = 5 - 3x^2 + 2x^3$ and x decreases at the rate of 6 units per seconds, how fast is the slope of the curve changes when $x = 7$.
2. If a particle is moving in a straight line such that the displacement s after time t is given by $s = \frac{1}{2}vt$, where v be the velocity of the particle. Prove that the acceleration a of the particle is constant.
3. The radius of the circle increases at the rate $0.4\,cm/\sec$. What is the increase of its circumference.
4. Find the rate of change per second of the volume of a ball with respect to its radius r when $r = 6\,cm$.
5. Find the rate of change per minute of the surface area of a ball with respect to its radius r when $r = 9\,m$.

6. Find the rate of change of the volume of the cone with respect to its height.
7. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x)=6x^3-27x^2+36x+6$.
8. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x)=-2x^3+6x^2+18x-1$.
9. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x)=\frac{\log x}{x}$ if $0<x<\infty$.
10. The maximum value of the function $y=-x^3+1$ over the interval [-2,1]
    b. 9 b. 0 c. 2 d. 10
11. Find the length (l) and breadth (b) of a rectangle with perimeter 12 such that it has maximum area.
    a. 3,3 b. 2,4 c. 3,4 d. 5,2

ANSWERS

1. -468
3. $0.8\pi\, cm/\sec$
4. $144\pi\, cm^3/\sec$
5. $72\pi\, m^2/\min$
6. $\frac{1}{3}\pi r^2$
7. $x=1$ is a point of maxima and maximum value of function is 21, $x=2$ is a point of minima and minimum value of function is 18.
8. $x=-1$ is a point of minima minimum value of function is -11, $x=3$ is a point of maxima and maximum value of function is 53.
9. $x=e$ is a point of maxima and maximum value of function is $\frac{1}{e}$.
10. b
11. a

UNIT – 5

INTEGRAL CALCULUS

Learning Objectives

- To learn the concept of integration and its geometrical meaning.
- To learn various formulae to evaluate the integrals.
- To learn about definite integral and its application to calculate the area under the curves.

5.1 INTEGRATION – Reverse operation of differentiation

Integration is the reverse process of differentiation. In the previous chapter, we have studied differentiation as the study of small change in one variable with respect to small change in other. In the same manner, integration is the study of a function as a whole when small changes are given. For example

if dA shows small change in area, then $\int dA$ is the area as a whole.

If f(x) and g(x) are two functions such that $\frac{dg(x)}{dx} = f(x)$, then $\int f(x)dx = g(x) + c$ i.e. g(x) is integral of f(x) with respect to x and c is constant of integration.

The function to be integrated is called integrand and put in between the sign $\int dx$.

Main Rule of Integration: We can integrate a function if it is in single form (variables/functions are not in product or quotient form) otherwise we will have to use various different methods to convert it in single form and then integrate.

5.2 SIMPLE STANDARD INTEGRAL

(a)Integral of algebraic functions:

$$\int 0\, dx = \text{ constant}$$

$$\int 1dx = x + c$$

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1 \text{ , is any real number } (n \neq -1)$$

$$\int \frac{1}{x} dx = \log x + c$$

Some Results of Integration

I. $\frac{d}{dx}\int (f(x)dx) = f(x)$

II. $\int kf(x)\,dx = k\int f(x)dx$ where k is any constant.

III. $\int (f(x) \pm g(x))dx = \int f(x)\,dx \pm \int g(x)\,dx$

IV. $\int kdx = kx + c$ where k is any constant and c constant of integration

Example 1. Evaluate (i) $\int x^3dx$ (ii) $\int \sqrt{x}\,dx$ (iii) $\int \frac{1}{\sqrt{x}}dx$ (iv) $\int 2dx$

(v) $\int (2x^2+3x+5)dx$ (vi) $\int (x^3 - 5x + \frac{1}{x})dx$

Sol : (i) $\int x^3dx = \frac{x^4}{4} + c$ (iv) $\int 2dx = 2x + c$

(ii) $\int \sqrt{x}\,dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$ (v) $\int (2x^2+3x+5)dx = 2\frac{x^3}{3} + 3\frac{x^2}{2} + 5x + c$

(iv) $\int \frac{1}{\sqrt{x}}dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$ (vi) $\int (x^3 - 5x + \frac{1}{x})dx = \frac{x^4}{4} - 5\frac{x^2}{2} + \log x + c$

Example 2. Evaluate (i) $\int (x+1)x^2\,dx$ (ii) $\int \frac{x^2+1}{x}dx$ (iii) $\int \frac{x^3 - x + 3}{x}dx$

(iv) $\int (\sqrt{x} + \frac{1}{\sqrt{x}})dx$

Sol : (i) $\int (x+1)x^2dx = \int (x^3 + x^2)dx = \frac{x^4}{4} + \frac{x^3}{3} + c$

(ii) $\int \frac{x^2+1}{x}dx = \int \left(x + \frac{1}{x}\right)dx = \frac{x^2}{2} + \log x + c$

(iii) $\int \frac{x^3 - x + 3}{x}dx = \int (x^2 - 1 + \frac{3}{x})dx = \frac{x^3}{3} - x + \log x + c$

(iv) $\int (\sqrt{x}+\frac{1}{\sqrt{x}})dx = \frac{x^{3/2}}{3/2} - \frac{x^{1/2}}{1/2} + c$

EXERCISE-I

Integrate the following functions with respect to x

1. Evaluate

   i. $\int x^4 dx$ ii. $\int x^{\frac{5}{4}} dx$ iii. $\int \frac{1}{x^5} dx$ iv. $\int \frac{1}{x^{3/4}} dx$

2. $\int (3x\sqrt{x}+4\sqrt{x}+5)dx$

3. $\int (\sqrt{x}+\frac{1}{\sqrt{x}})^2 dx$

4. $\int \frac{x^3+5x^2+4x+1}{x^2} dx$

5. $\int \left(x+\frac{1}{x}\right)^3 dx$

6. $\int \frac{(1+x)^2}{\sqrt{x}} dx$

7. $\int \frac{(x+1)(x-2)}{\sqrt{x}} dx$

ANSWERS

1. (i) $\frac{x^5}{5}+c$ (ii) $\frac{4}{9}x^{\frac{9}{4}}+c$ (iii) $\frac{x^{-4}}{-4}+c$ (iv) $4x^{\frac{1}{4}}+c$

2. $\frac{6}{5}x^{\frac{5}{2}}+\frac{8}{3}x^{\frac{3}{2}}+5x+c$

3. $\frac{x^2}{2}+\log x+2x+c$

4. $\frac{x^2}{2}+5x+4\log x-\frac{1}{x}+c$

5. $\frac{x^4}{4}-\frac{1}{2x^2}+3\frac{x^2}{2}+3\log x+c$

6. $2x^{\frac{1}{2}}+\frac{2}{5}x^{\frac{5}{2}}+\frac{4}{3}x^{\frac{3}{2}}+c$

7. $\frac{2}{5}x^{\frac{5}{2}}-\frac{2}{3}x^{\frac{3}{2}}-4x^{\frac{1}{2}}+c$

(b) Integrals of the type $(ax + b)^n$

When $n \neq -1$, $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a} + c, n \neq -1$, n is any real number

When $n = 1$, $\int \frac{1}{ax+b} dx = \frac{\log(ax+b)}{a} + c$

Example 3. Evaluate (i) $\int (1-3x)^5 x\, dx$ (ii) $\int \sqrt{2x+3}\, dx$

(iii) $\int \frac{1}{3+2x} dx$ (iv) $\int \frac{1}{(5+3x)^3} dx$

Sol : (i) $\int (1-3x)^5 dx = \frac{(1-3x)^6}{6(-3)} + c = \frac{(1-3x)^6}{-18} + c$

(ii) $\int \sqrt{2x+3}\, dx = \frac{(2x+3)^{3/2}}{\frac{3}{2} \times 2} + c = \frac{(2x+3)^{3/2}}{3} + c$

(iii) $\int \frac{1}{3+2x} dx = \frac{1}{2} \log(3+2x) + c$

(iv) $\int \frac{1}{(5+3x)^3} dx = \int (5+3x)^{-3} dx = \frac{(5+3x)^{-2}}{-2 \times 3} + c = \frac{(5+3x)^{-2}}{-6} + c$

EXERCISE-II

Integrate the following functions with respect to x

1. $\int \frac{x^3 - x + 3}{x+1} dx$

2. $\int \sqrt{3-2x}\, dx$

3. $\int (5+3x)^{-7}\,dx$

4. $\int (5x+3)^{4}\,dx$

5. $\int \frac{1}{\sqrt{1-4x}}\,dx$

6. $\int \frac{1}{\sqrt{2x+5}+\sqrt{2x-5}}\,dx$

7. $\int \frac{x^2+5x+2}{x+2}\,dx$

8. $\int \left(\frac{1}{2-3x}+\frac{1}{\sqrt{3x-2}}\,dx\right)$

9. $\int \sqrt{3+2x}+(3x-5)^4\,dx$

10. $\int \frac{x-1}{\sqrt{x+4}}\,dx$

ANSWERS

1. $\frac{x^3}{3}-\frac{x^2}{2}+3\log(x+1)+c$

2. $\frac{(3-2x)^{3/2}}{-3}+c$

3. $\frac{(5+3x)^{-8}}{-24}+c$

4. $\frac{(5x+3)^5}{25}+c$

5. $\frac{(1-4x)^{1/2}}{-2}+c$

6. $\frac{1}{30}[(2x+5)^{3/2}-(2x-5)^{3/2}]+c$

7. $\frac{x^2}{2}+3x-4\log(x+2)+c$

8. $\frac{\log(2-3x)}{-3}+\frac{2(3x-2)^{1/2}}{3}+c$

9. $\frac{(3+2x)^{3/2}}{3}+\frac{(3x-5)^5}{15}+c$

10. $\frac{2(x+4)^{3/2}}{3}-10(x+4)^{1/2}+c$

(c) More formulae of integrals of the following type:

1. $\int \frac{1}{x^2+a^2}\,dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$

2. $\int \frac{-1}{x^2+a^2}\,dx=\frac{1}{a}\cot^{-1}\frac{x}{a}+c$

3. $\int \frac{1}{\sqrt{x^2+a^2}}dx = \log|x+\sqrt{x^2+a^2}|+c$

4. $\int \frac{1}{\sqrt{x^2-a^2}}dx = \log|x+\sqrt{x^2-a^2}|+c$

5. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}+c$

6. $\int \frac{-1}{\sqrt{a^2-x^2}}dx = \cos^{-1}\frac{x}{a}+c$

7. $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\frac{x-a}{x+a}+c$

8. $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\frac{a+x}{a-x}+c$

9. $\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\frac{x}{a}+c$

10. $\int \frac{-1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a}+c$

Example 4. Evaluate (i) $\int \frac{dx}{x^2+9}$ (ii) $\int \frac{dx}{2x^2+9}$ (iii) $\int \frac{dx}{\sqrt{1-3x^2}}$

(iv) $\int \frac{dx}{\sqrt{4x^2+16}}$ (v) $\int \frac{dx}{x^2-25}$ (vi) $\int \frac{dx}{9x^2-16}$

Sol : (i) $\int \frac{dx}{x^2+9} = \int \frac{dx}{x^2+3^2}$

Using formula $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$

Here a = 3

So $\int \frac{dx}{x^2+3^2} = \frac{1}{3}\tan^{-1}\frac{x}{3}+c$

(ii) $\int \frac{dx}{2x^2+9} = \frac{1}{2}\int \frac{dx}{\frac{2x^2+9}{2}} = \frac{1}{2}\int \frac{dx}{x^2+\frac{9}{2}} = \frac{1}{2}\int \frac{dx}{x^2+\left(\frac{3}{\sqrt{2}}\right)^2}$

Using formula $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

$$a = \frac{3}{\sqrt{2}}$$

$$\therefore \qquad \frac{1}{2}\int \frac{dx}{x^2+\left(\frac{3}{\sqrt{2}}\right)^2} = \frac{1}{2}\left[\frac{1}{3/\sqrt{2}}\tan^{-1}\frac{x}{3/\sqrt{2}}\right] + c$$

$$= \frac{1}{3\sqrt{2}}\tan^{-1}\frac{\sqrt{2}x}{3} + c$$

(iii) $$\int \frac{dx}{\sqrt{1-9x^2}} = \int \frac{dx}{\sqrt{\left(\frac{1-9x^2}{9}\right)9}} = \int \frac{dx}{3\sqrt{\frac{1}{9}-x^2}}$$

$$= \frac{1}{3}\int \frac{dx}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}} \qquad \text{Here } a = \frac{1}{3}$$

Using formula $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

$$\frac{1}{3}\int \frac{dx}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}} = \frac{1}{3}[\sin^{-1}\frac{x}{1/3}] + c = \frac{1}{3}[\sin^{-1}3x] + c$$

(iv) $$\int \frac{dx}{\sqrt{4x^2+16}} = \int \frac{dx}{\sqrt{\left(\frac{4x^2+16}{4}\right)4}}$$

$$= \int \frac{dx}{2\sqrt{x^2+4}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2+2^2}}$$

Here a = 2. Using formula $\int \frac{dx}{\sqrt{x^2+a^2}} = \log|x+\sqrt{x^2+a^2}| + c$

$$\frac{1}{2}\int \frac{dx}{\sqrt{x^2+2^2}} = \frac{1}{2}[\log|x+\sqrt{x^2+2^2}|] + c$$

(v) $$\int \frac{dx}{\sqrt{x^2-25}} = \int \frac{dx}{\sqrt{x^2-5^2}}$$

Using formula $\int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}|+c$

$$\therefore \qquad \frac{dx}{\sqrt{x^2-5^2}} = \log|x+\sqrt{x^2-5^2}|+c$$

(vi) $$\int \frac{dx}{9x^2-16} = \int \frac{dx}{\frac{9x^2-16}{9}\times 9} = \frac{1}{9}\int \frac{dx}{x^2-\frac{16}{9}}$$

$$\frac{1}{9}\int \frac{dx}{x^2-\left(\frac{4}{3}\right)^2}. \quad \text{Here } a = \frac{4}{3}$$

Using formula $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}+c$

$$\therefore \qquad \frac{1}{9}\int \frac{dx}{x^2-\left(\frac{4}{3}\right)^2} = \frac{1}{9}\left[\frac{1}{2\times\frac{4}{3}}\log\frac{x-\frac{4}{3}}{x+\frac{4}{3}}\right]+c$$

$$= \frac{1}{9}\left[\frac{3}{8}\log\frac{3x-4}{3x+4}\right]+c$$

$$= \frac{1}{24}\log\frac{3x-4}{3x+4}+c$$

EXERCISE-III

1. Integrate the following integral with respect to x.

(i) $\int \frac{dx}{\sqrt{9-x^2}}$ (ii) $\int \frac{dx}{x\sqrt{x^2-25}}$ (iii) $\int \frac{dx}{\sqrt{16x^2-36}}$

(iv) $\int \frac{dx}{25x^2-16}$ (v) $\int \frac{dx}{x^2+49}$ (vi) $\int \frac{dx}{4+9x^2}$

2. Evaluate $\int \frac{x^2}{1+x^2}dx$

3. Integrate $\dfrac{x^6+2}{x^2+1}$ with respect to x.

4. Integrate $\left(1+\dfrac{1}{1+x^2}-\dfrac{2}{\sqrt{1-x^2}}+\dfrac{5}{x\sqrt{x^2-1}}\right)$ with respect to x.

ANSWERS

1. (i) $\sin^{-1}\dfrac{x}{3}+c$ (ii) $\dfrac{1}{5}\sec^{-1}\dfrac{x}{5}+c$ (iii) $\dfrac{1}{4}\log\left|x+\sqrt{x^2-\left(\dfrac{3}{2}\right)^2}\right|+c$

(iv) $\dfrac{1}{40}\log\dfrac{5x-4}{5x+4}+c$ (v) $\dfrac{1}{7}\tan^{-1}\dfrac{x}{7}+c$ (vi) $\dfrac{1}{6}\tan^{-1}\dfrac{3x}{2}+c$

2. $x-\tan^{-1}x+c$ 3. $\dfrac{x^5}{5}-\dfrac{x^3}{3}+x+\tan^{-1}x+c$

4. $x+\tan^{-1}x-2\sin^{-1}x+5\sec^{-1}x+c$

(c) Integrals of the Exponential functions

An exponential function is of the form (constant)$^{\text{variable}}$, i.e. e^x, a^x, b^{3x} etc.

(i) $\displaystyle\int a^{mx}\,dx=\frac{a^{mx}}{(\log a)m}+c$

(ii) $\displaystyle\int e^{mx}dx=\frac{e^{mx}}{m}+c$

Example 5. Evaluate (i) $\int 3^x\,dx$ (ii) $\displaystyle\int\frac{a^x}{b^x}dx$

(iii) $\displaystyle\int\frac{a^x+b^x}{a^xb^x}dx$ (iv) $\displaystyle\int\frac{2^{3x}b^x}{e^x}\,dx$

Sol : (i) $\displaystyle\int 3^x\,dx=\frac{3^x}{(\log 3).1}+c$

(ii) $\displaystyle\int \frac{a^x}{b^x}dx = \int \left(\frac{a}{b}\right)^x dx = \frac{(a/b)^x}{\log(a/b)} + c$

(iii) $\displaystyle\int \frac{a^x + b^x}{a^x b^x}dx = \int \left(\frac{a^x}{a^x b^x} + \frac{b^x}{a^x b^x}\right)dx$

$$= \int \left(\frac{1}{b^x} + \frac{1}{a^x}\right)dx$$

$$= \int (b^{-x} + a^{-x})dx$$

$$= \frac{b^{-x}}{(\log b)(-1)} + \frac{a^{-x}}{(\log a)(-1)} + c$$

(iv) $\displaystyle\int \frac{2^{3x} b^x}{e^x}dx = \int \left(\frac{2^3 b}{e}\right)^x dx = \int \left(\frac{8b}{e}\right)^x dx$

$$= \frac{\left(\frac{8b}{e}\right)^x}{\log\left(\frac{8b}{e}\right).1} + c$$

EXERCISE -IV

1. Evaluate $\int (a^{2x} + b^{2x})dx$.

2. Integrate $\dfrac{(a^x + b^x)^2}{a^x b^x}$ with respect to x.

3. Integrate $\left(\dfrac{1}{b^x} - \dfrac{1}{a^x}\right)$ with respect to x.

4. Evaluate $\int (x^a + e^x + e^a)dx$.

5. Integrate $\dfrac{2^x + 3^x}{5^x}$ with respect to x.

ANSWERS

1. $\dfrac{a^{2x}}{(\log a)2}+\dfrac{b^{2x}}{(\log b)2}+c$

2. $\dfrac{\left(\frac{a}{b}\right)^x}{\log\frac{a}{b}}+\dfrac{\left(\frac{b}{a}\right)^x}{\log\frac{b}{a}}+2x+c$

3. $-\dfrac{b^{-x}}{(\log b)}+\dfrac{a^{-x}}{(\log a)}+c$

4. $\dfrac{x^{a+1}}{a+1}+\dfrac{e^x}{1}+e^a x+c$

5. $\dfrac{\left(\frac{2}{5}\right)^x}{\log\frac{2}{5}}+\dfrac{\left(\frac{3}{5}\right)^x}{\log\frac{3}{5}}+c$

(d) Integral of Trigonometric Functions

1. $\int \sin x\,dx = -\cos x + c$
2. $\int \cos x\,dx = \sin x + c$
3. $\int \tan x\,dx = \log \sec x + c$ or $-\log \cos x + c$
4. $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + c$
5. $\int \sec x\,dx = \log|\sec x + \tan x| + c$
6. $\int \cot x\,dx = \log \sin x + c$ or $-\log \operatorname{cosec} x + c$
7. $\int \sec^2 x\,dx = \tan x + c$
8. $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$
9. $\int \sec x \tan x\,dx = \sec x + c$
10. $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$

Example 6. Evaluate : (i) $\int \tan^2 x\,dx$ (ii) $\int \cot^2 x\,dx$

(iii) $\int \sin 3x \cos 2x\,dx$ (iv) $\int \dfrac{2+3\cos x}{\sin^2 x}dx$

(v) $\int \dfrac{1}{1+\sin x}dx$ (vi) $\int \dfrac{dx}{1-\cos x}$

Sol : (i) $\int \tan^2 x\,dx = \int (\sec^2 x - 1)dx = \int \sec^2 x\,dx - \int dx$

$$= \tan x - x + c$$

(ii) $\int \cot^2 x\,dx = \int (\operatorname{cosec}^2 x - 1)dx$

$$= \int \operatorname{cosec}^2 x\,dx - \int dx = -\cot x - x + c$$

(iii) $\int \sin 3x \cos 2x dx = \frac{1}{2}\int 2\sin 3x \cos 2x dx$

Using formula 2 sin A cos B = sin (A + B) + sin (A – B)

$$= \frac{1}{2}\int [\sin(3x+2x) + \sin(3x-2x)]dx$$

$$= \frac{1}{2}\int (\sin 5x + \sin x)dx$$

$$= \frac{1}{2}\left[-\frac{\cos 5x}{5} - \frac{\sin x}{1}\right] + c$$

(iv) $\int \frac{2+3\cos x}{\sin^2 x}dx$

$$= \int \frac{2}{\sin^2 x}dx + \int \frac{3\cos x}{\sin^2 x}dx$$

$$= 2\int \operatorname{cosec}^2 x\,dx + 3\int \cot x \operatorname{cosec} x\,dx$$

$$= -2\cot x - 3\operatorname{cosec} x + c$$

(v) $\int \frac{dx}{1+\sin x}$

$$= \int \frac{dx}{1+\sin} \times \frac{1-\sin x}{1-\sin x}$$

$$= \int \frac{1-\sin x}{1-\sin^2 x}dx \qquad \text{as } (a-b)(a+b) = a^2 - b^2$$

$$= \int \frac{1-\sin x}{\cos^2 x}dx$$

$$= \int \frac{1}{\cos^2 x}dx - \int \frac{\sin x}{\cos^2 x}dx$$

$$= \int \sec^2 x dx - \int \tan x \sec x \, dx$$

$$= \tan x - \sec x + c$$

(vi) $\int \frac{dx}{1-\cos x}$

$$= \int \frac{1}{1-\cos x} \times \frac{1+\cos x}{1+\cos x} dx$$

$$= \int \frac{1+\cos x}{1-\cos^2 x} dx$$

$$= \int \frac{1+\cos x}{\sin^2 x} dx$$

$$= \int \frac{1}{\sin^2 x} dx + \int \frac{\cos x}{\sin^2 x} dx$$

$$= \int \operatorname{cosec}^2 x \, dx + \int \cot x \operatorname{cosec} x \, dx$$

$$= -\cot x - \operatorname{cosec} x + c$$

EXERCISE–V

1. Find the value of the integral $\int \frac{dx}{x^2+9}$ is ……............

   a. $tan^{-1}\left(\frac{x}{3}\right) + c$ c. $-\frac{1}{3}cot^{-1}\left(\frac{x}{9}\right) + c$

   b. $\frac{1}{9}tan^{-1}\left(\frac{x}{9}\right) + c$ d. $\frac{1}{3}tan^{-1}\left(\frac{x}{3}\right) + c$

2. The value of $\int \frac{x}{x^2+1} dx$ is ....

   a. $\frac{1}{2}\log\left(x^2+1\right) + c$ b. $\log\left(x^2+1\right) + c$ c. $tanx + c$ d. $cotx + c$

3. Evaluate $\int (3\sin x - 4\cos x + \frac{5}{\cos^2 x} - \frac{6}{\sin^2 x} + \sec x) dx$.

4. Evaluate $\int (\sec^2 x + \operatorname{cosec}^2 x) dx$.

5. Evaluate $\int \frac{5\cos^2 x + 6\sin^2 x}{2\sin^2 x \cos^2 x} dx$.

6. Evaluate $\int (\tan x + \cot x)^2 dx$.

7. Evaluate $\int \frac{dx}{1+\cos x}$.

8. Evaluate $\int \frac{dx}{1-\sin x}$.

9. Evaluate $\int 2\cos 3x \cos x\, dx$.

10. Evaluate $\int \sin 5x \sin 2x\, dx$.

11. Evaluate $\int \frac{\sin x}{1+\sin x} dx$.

12. Evaluate $\int \frac{\sec x}{\sec x+\tan x} dx$.

ANSWERS

1. (a) 2. (a)

3. $-3\cos x - 4\sin x + 5\tan x + 6\cot x + \log|\sec x + \tan x| + c$

4. $\tan x - \cot x + c$

5. $-\frac{5}{2}\cot x + 3\tan x + c$

6. $\tan x - \cot x + c$

7. $-\cot x + \text{cosec}\, x + c$

8. $\tan x + \sec x + c$

9. $\frac{\sin 4x}{4} + \frac{\sin 2x}{2} + c$

10. $\frac{1}{2}\left[-\frac{\sin 7x}{7} + \frac{\sin 3x}{3}\right] + c$

11. $x - \tan x + \sec x + c$

12. $\tan x - \sec x + c$

There are various methods to convert the product/quotient of functions into a single function like Multiple/divide/splitting/Use of formulae. If a given function can't be integrated by the methods explained till now :-

(1) Integration by parts

(2) Substitution method

(3) Method of partial fraction

Integration by Parts Method

If f(x) and g(x) are two functions of x then

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int \left\{\frac{d}{dx}f(x)\int g(x)dx\right\}dx .$$

Here in above formula f(x) is chosen as first function and g(x) is chosen as second function.

Note : Selection of I function and II function in accordance with function which comes first in the word "ILATE".

where I = Inverse functions

L = Logarithmic functions

A = Algebraic functions

T = Trigonometric functions

E = Exponential functions

Example 7. Evaluate (i) $\int x\sin x\,dx$ (ii) $\int x\tan^2 x\,dx$ (iii) $\int \log x\,dx$

Sol : (i) $\int x\sin x\,dx = x\int \sin x\,dx - \int \left\{\frac{d}{dx}x\int \sin x dx\right\}dx$

$$= x(-\cos x) - \int 1\{-\cos x\}dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + c$$

(ii) $\int x\tan^2 x\,dx = \int x(\sec^2 x - 1)dx$

$$= \int x\sec^2 x\,dx - \int x\,dx$$

$$= x\int \sec^2 x\,dx - \int \left\{\frac{d}{dx}x\int \sec^2 x\,dx\right\}dx - \frac{x^2}{2}$$

$$= x\tan x - \int \{1\tan x\}dx - \frac{x^2}{2}$$

$$= x\tan x - \int \tan x\,dx - \frac{x^2}{2}$$

$$= x\tan x - \log\sec x - \frac{x^2}{2} + c$$

(iii) $\int \log x \, dx = \int (1 \log x) dx$

$$= \int x^0 \log x \, dx$$

$$= \log x \int x^0 dx - \int \left\{ \frac{d}{dx} \log x \int x^0 dx \right\} dx$$

$$= (\log x)x - \int \left(\frac{1}{x} x \right) dx$$

$$= (\log x)x - \int 1 dx$$

$$= (\log x)x - x + c$$

Example 8. Evaluate (i) $\int x^2 \cos x \, dx$ (ii) $\int x^2 e^x \, dx$.

Sol : (i) $\int x^2 \cos x \, dx = x^2 \int \cos x \, dx - \int \left\{ \frac{d}{dx} x^2 \int \cos x \, dx \right\} dx$

$$= x^2 \sin x - \int 2\{x \sin x\} dx$$

$$= x^2 \sin x - 2\int x \sin x \, dx$$

Again by parts method

$$= x^2 \sin x - 2\left[x \int \sin x dx - \int \left\{ \frac{d}{dx} x \int \sin x \, dx \right\} dx \right]$$

$$= x^2 \sin x - 2\left[x(-\cos x) - \int \{1 \times -\cos x\} dx \right]$$

$$= x^2 \sin x - 2\left[-x \cos x + \int \cos x \, dx \right]$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + c$$

(ii) $\int x^2 e^x dx = x^2 \int e^x dx - \int \left\{ \frac{d}{dx} x^2 \int e^x \, dx \right\} dx$

$$= x^2 e^x - \int \left\{ 2x \, e^x \right\} dx$$

$$= x^2 e^x - 2\int x \, e^x \, dx$$

Again by parts method

$$= x^2e^x - 2\left[x\int e^x\,dx - \int \left\{\frac{d}{dx}x\int e^x\,dx\right\}dx\right]$$

$$= x^2e^x - 2\left[xe^x - \int (1e^x)dx\right]$$

$$= x^2e^x - 2xe^x + e^x + c$$

EXERCISE -VI

1. $\int x\cos x\,dx$
2. $\int \log(x+1)dx$
3. $\int x\,e^x\,dx$
4. $\int x^2\log x\,dx$
5. $\int x\cot^2 x\,dx$
6. $\int x^2\cos 2x dx$
7. $\int x^2e^{-x}\,dx$
8. $\int x\,cosec^2 x\,dx$

ANSWERS

1. x sin x + cos x + c
2. x log (x + 1) – x + log (x + 1) + c
3. $xe^x - e^x + c$
4. $\frac{x^3}{3}\log x - \frac{1}{9}x^3 + c$
5. $-x\cot x + \log\sin x - \frac{x^2}{2} + c$
6. $\frac{x^2}{2}\sin 2x + \frac{x}{2}\cos 2x - \frac{\sin 2x}{4} + c$
7. $-x^2e^{-x} - 2xe^{-x} - 2e^{-x} + c$
8. $-x\cot x + \log\sin x + c$

5.3EVALUATION OF DEFINITE INTEGRALS

Introduction

If f(x) is a continuous function defined on closed interval [a, b] and F(x) is the integral of f(x) i.e.

$$\int f(x)\,dx = F(x)$$

then definite integral of f(x) in closed interval [a, b] is

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

where a is lower limit, b is upper limit.

Example 9. Evaluate (i) $\int_1^2 x^2\,dx$ (ii) $\int_0^{\pi/4} \sin x\,dx$ (iii) $\int_0^1 e^x\,dx$

(iv) $\int_4^9 \sqrt{x}\,dx$ (v) $\int_1^2 \frac{1}{x}\,dx$

Sol : (i) $\int_1^2 x^2 dx = \left[\frac{x^3}{3}\right]_1^2 = \left(\frac{2^3}{3} - \frac{1^3}{3}\right) = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$.

(ii) $\int_0^{\pi/4} \sin x\,dx = [-\cos x]_0^{\pi/4} = \left(-\cos\frac{\pi}{4}\right) - (-\cos 0) = \left(-\frac{1}{\sqrt{2}} + 1\right)$

(iii) $\int_0^1 e^x\,dx = \left[\frac{e^x}{1}\right]_0^1 = (e^1 - e^0) = (e - 1)$

(iv) $\int_4^9 \sqrt{x}\,dx = \left[\frac{x^{3/2}}{3/2}\right]_4^9 = \frac{2}{3}\left[x^{3/2}\right]_4^9 = \frac{2}{3}[9^{3/2} - 4^{3/2}]$

$$= \frac{2}{3}[27 - 8] = 2 \times \frac{19}{3} = \frac{38}{3}$$

(v) $\int_1^2 \frac{1}{x}\,dx = [\log x]_1^2 = \log 2 - \log 1 = \log 2$ as (log 1 = 0)

Example 10. Evaluate (i) $\int_0^1 \frac{dx}{1+x^2}$ (ii) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ (iii) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Sol : (i) $\int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0$

$$= \tan^{-1}\tan\frac{\pi}{4} - \tan^{-1}\tan 0 = \left(\frac{\pi}{4} - 0\right) = \frac{\pi}{4}.$$

(ii) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}} = [\sec^{-1} x]_1^2$

$$= \sec^{-1} 2 - \sec^{-1} 1$$

$$= \sec^{-1}\sec\frac{\pi}{3} - \sec^{-1}\sec 0$$

$$= \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

EXERCISE -VII

1. Find the value of the integral $\int_0^{\frac{\pi}{4}} tanx\, dx$.

   a. $\log 2$ b. $\frac{1}{2}$ c. $\log\sqrt{2}$ d. 2

2. Evaluate the integral $\int_{-2}^{2} e^x dx$

   a. $e^2 - e^{-2}$ b. $e^{-2} - e^2$ c. $e^{-2} + e^2$ d. $2e^2$

3. Evaluate : (i) $\int_2^3 \frac{dx}{x}$ (ii) $\int_{-5}^{5} x\,dx$ (iii) $\int_1^2 \frac{dx}{1+3x}$

(iv) $\int_2^3 (x^2+1)dx$ (v) $\int_{-2}^{2} e^{-x}\,dx$ (vi) $\int_0^{\pi/3} \cos x\,dx$

(vii) $\int_0^2 \frac{dx}{4+3x}$ (viii) $\int_0^2 \frac{dx}{x^2+4}$ (ix) $\int_0^{\pi/4} \tan x dx$

ANSWERS

1. (c) 2. (a)

3. (i) $\log\frac{3}{2}$ (ii) 0 (iii) $\frac{1}{3}\log\frac{7}{4}$ (iv) $\frac{22}{3}$ (v) $-e^{-2} + e^2$

(vi) $\frac{\sqrt{3}}{2}$ (vii) $\frac{1}{3}\log\frac{5}{2}$ (viii) $\frac{\pi}{8}$ (ix) $\log\sqrt{2}$.

Integral of the type when limit is 0 to $\frac{\pi}{2}$

$$\int_0^{\pi/2} \sin^n x\,dx \quad \text{or} \quad \int_0^{\pi/2} \cos^n x\,dx \quad \text{or} \quad \int_0^{\pi/2} \sin^n x \cos^m x\,dx$$

where n, m are natural numbers, we have

(i) $$\int_0^{\frac{\pi}{2}} \sin^x x dx = \begin{cases} \frac{(n-1)(n-3).....3.1}{n(n-2)(n-4).....4.2} \times \frac{\pi}{2} & if\ n = even \\ \frac{(n-1)(n-3)......4.2}{n(n-2)\quad ......5.3} & if\ n = odd \end{cases}$$

(ii) $$\int_0^{\frac{\pi}{2}} \cos^n x = \begin{cases} \frac{(n-1)(n-3).......3.1}{n(n-2)(n-4).....4.2} \times \frac{\pi}{2} & if\ n = even \\ \frac{(n-1)(n-3)......4.2}{n(n-2)\quad ......5.3} & if\ n = odd \end{cases}$$

(iii) $$\int_0^{\pi/2} \sin^n x \cos^m x\, dx = \frac{(n-1)(n-3)...(m-1)(m-3)...}{(n+m)(n+m-2)(n+m-4)...} \ (\times \frac{\pi}{2} \text{ if n and m both are even})$$

Note : Above formulae are called 'Reduction Formulae'

Example 11. Evaluate (i) $\int_0^{\pi/2} \sin^5 x\, dx$ (ii) $\int_0^{\pi/2} \cos^4 x\, dx$ (iii) $\int_0^{\pi/2} \sin^3 x \cos^2 x\, dx$

Sol : (i) $\int_0^{\pi/2} \sin^5 x\, dx = \frac{4\times 2}{5\times 3\times 1} = \frac{8}{15}$

(ii) $\int_0^{\pi/2} \cos^4 x\, dx = \frac{3\times 1}{4\times 2} \times \frac{\pi}{2} = \frac{3\pi}{16}$

Here n = 4(even)

(iii) $\int_0^{\pi/2} \sin^3 x \cos^2 x\, dx = \frac{2\times 1}{5\times 3\times 1} = \frac{2}{15}$

EXERCISE-VIII

Evaluate the integral :

1. $\int_0^{\pi/2} \sin^8 x\, dx$
2. $\int_0^{\pi/2} \cos^5 x\, dx$
3. $\int_0^{\pi/2} \sin^4 x\, dx$
4. $\int_0^{\pi/2} \cos^6 x\, dx$
5. $\int_0^{\pi/2} \cos^3 x \sin^4 x\, dx$
6. $\int_0^{\pi/2} \cos^6 x \sin^4 x\, dx$

7. Evaluate $I = \dfrac{\int_0^{\pi/2} \sin^3 x \cos^5 x \, dx}{\int_0^{\pi/2} \cos^4 x \, dx}$

ANSWERS

1. $\dfrac{35\pi}{256}$ 2. $\dfrac{8}{15}$ 3. $\dfrac{3\pi}{16}$ 4. $\dfrac{5\pi}{32}$ 5. $\dfrac{2}{35}$

6. $\dfrac{3\pi}{512}$ 7. $\dfrac{2}{9\pi}$

5.4 APPLICATIONS OF INTEGRATION

There are many applications that requires integration techniques like Area under the curve, volumes of solid generated by revolving the curve about axes, velocity, acceleration, displacement, work done, average value of function. But in this chapter, we will be taking a look at couple of applications of integrals.

(i) If y = f(x) is any function of x, then

Area under the curve is $\int_{x=a}^{x=b} y \, dx$

(ii) If v = f(t) is velocity of particle at time t then,

Displacement S = $\int_{t=t_1}^{t=t_2} v \, dt$

Example 12. Find the area under the curve $y = 1 + 2x^3$, when $0 \le x \le 2$.

Sol : Given $y = 1 + 2x^3$, a = 0, b = 2.

$$\text{Area} = \int_a^b y \, dx = \int_0^2 (1 + 2x^3) dx$$

$$= \left[x + \frac{x^4}{2} \right]_0^2 = (2+8) - (0+0) = 10 \text{ units}$$

Example 13. Find area under the curve of y = sin x, when $0 \le x \le \dfrac{\pi}{2}$.

Sol : Equation of curve y = sin x, a = 0, b = $\frac{\pi}{2}$

$$\text{Area} = \int_a^b y\,dx = \int_0^{\pi/2} \sin x\,dx = [-\cos x]_0^{\pi/2}$$

$$= [-\cos\frac{\pi}{2}] - [-\cos 0]$$

$$= [-0] + 1 = 1 \quad \text{Square units}$$

EXERCISE-IX

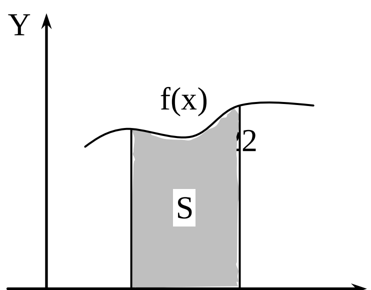
1. Find the area under the curve $y = 4x^2$, when $0 \le x \le 3$.
2. Find the area under the curve $y = \frac{1}{x}$, when $2 \le x \le 4$.
3. Find the area under the curve y = cos x, when $0 \le x ,\ \frac{\pi}{4}$.
4. Find the area under the curve $y = e^{2x}$, when $0 \le x \le 1$.
5. Find area under the curve $y = \sqrt{2x+3}$, when $3 \le x \le 11$.

ANSWERS

1. 36 2. log 2 3. $\frac{1}{\sqrt{2}}$ 4. $\frac{e^2-1}{2}$ 5. $\frac{98}{3}$

5.5NUMERICAL INTEGRATION OR APPROXIMATE INTEGRATION

Numerical integration is an approximate solution to a definite integral $\int_a^b f(x)\,dx$. In this chapter we are taking two methods to find approximate value of definite integral, $\int_a^b f(x)\,dx$ as area under the curve when $a \le x \le b$.



(i) Trapezoidal Rule

(ii) Simpson's Rule

(i) Trapezoidal Rule : If y = f(x) be the equation of curve, then approximate area under the curve y = f(x), when $a \le x \le b$ is

$$\int_a^b y\,dx = \frac{h}{2}[(\text{first ordinate + last ordinate}) + 2\,(\text{sum of remaining ordinates})]$$

Here $h = \frac{b-a}{n}$ $\qquad$ $a \rightarrow$ lower limit of x

$b \rightarrow$ upper limit of x

$n \rightarrow$ Number of integrals

* Ordinates are always one more than the number of intervals.

In mathematics, and more specifically in Number analysis, the trapezoidal rule (also known as trapezoid rule or trapezium rule) is a technique, for approximating the definite integral $\int_a^b f(x)\,dx$. The trapezoidal rule works by approximating the region under the graph of function f(x) as a trapezoid and calculating its area.

Example 14. Find the approximate area under the curve using trapezoidal rule, determined by the data given below

x	0	1	2	3	4	5
y	0	2.5	3	4.5	5	7.5

Sol : Here equation of curve not given so let y = f(x)

Lower limit of x, i.e. a = 0

Upper limit of x , i.e. b = 5

Gap between values of x i.e. h = 1

Number of ordinates in table are = 6

So by trapezoidal Rule

$$\int_0^5 y\,dx = \frac{h}{2}[(y_1 + y_6) + 2(y_2 + y_3 + y_4 + y_5)]$$

$y_1 = 0$, $y = 2.\ 5$, $y_3 = 3$, $y_4 = 4.5$, $y_3 = 5$, $y_6 = 7.5$

$$= \frac{1}{2}[(0+7.5)+2(2.5+3+4.5+5)]$$

$$= \frac{1}{2}[7.5+30] = 18.75 \text{ square units}$$

Example 15. Using trapezoidal rule, evaluate $\int_0^{2.5}(1+x)\,dx$ by taking six ordinates.

Sol : Given y = 1 + x,a = 0, b = 2.5

as ordinates are 6 $\Rightarrow$ n = 5

$$h = \frac{b-a}{n} = \frac{2.5-0}{5} = 0.5$$

x	0	0.5	1	1.5	2	2.5
$y = 1+x$	1	1.5	2	2.5	3	3.5

Here $y_1 = 1$, $y_2 = 1.5$, $y_3 = 2$, $y_4 = 2.5$, $y_5 = 3$, $y_6 = 3.5$.

By trapezoidal Rule

$$\int_0^{2.5}(1+x)dx = \frac{h}{2}[(y_1 + y_6) + 2(y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{0.5}{2}[(1+3.5)+2(1.5+2+2.5+3)]$$

$$= \frac{1}{4}[4.5+18] = \frac{22.5}{4} = 5.6 \text{ square units}$$

EXERCISE -X

1. Using trapezoidal rule, evaluate $\int_0^2 \sqrt{(9-x^2)}\,dx$ by taking 4 equal intervals.

2. Using trapezoidal Rule, Find the approximate area under the curve $y = x^2 + 1$, when $0 \le x \le 6$ by taking 7 ordinates.

3. A curve is drawn to pass through the points given below

x	2	2.2	2.4	2.6	2.8	3.0
y	3	3.4	3.7	3.9	4	3.2

Find the approximate area bounded by the curve, x axis and the lines x = 2 and x = 3.

4. Apply Trapezoidal rule to evaluate $\int_4^8 \frac{1}{x+2}$ by taking 4 equal intervals.

ANSWERS

1. 5.4 2. 79 3. 3.62 4. 0.5123

Simpson's $\frac{1}{3}$rd Rule : In this rule, the graph of curve y = f(x) is divided into 2n (Even) intervals.

Let y = f(x) be the equation of curve. When $a \le x \le b$, then approximate area under the curve is

$$\int_a^b y\,dx = \frac{h}{3}\ [(\text{first ordinate + last ordinate}) + 2(\text{sum of remaining odd ordinates})$$

$$+ 4(\text{sum of remaining even ordinates})]$$

where $h = \frac{b-a}{n}$

$a \to$ lower limit of x

$b \to$ upper limit of x

$n \to$ number of intervals

Note : Simpson's Rule is valid when total number of ordinates are odd.

Example 16. Calculate by Simpson's rule an approximate value of $\int_2^8 (x^3+3)dx$ by taking 7 ordinates.

Sol : Given $y = x^3 + 3$, a = 2, b = 8, n = 6

$$\therefore \qquad h = \frac{b-a}{n} = \frac{8-2}{6} = 1$$

x	2	3	4	5	6	7	8
$y = x^3 + 3$	11	30	67	128	219	347	515

Here $y_1 = 11$, $y_2 = 30$, $y_3 = 67$, $y_4 = 128$, $y_5 = 219$, $y_6 = 347$, $y_7 = 515$.

By Simpson's rule

$$\int_2^8 (x^3+3)dx = \frac{h}{3}(y_1+y_7)+2(y_3+y_5)+4(y_2+y_4+y_6)]$$

$$= \frac{1}{3}[(11+515)+2(67+219)+4(30+128+347)]$$

$$= \frac{1}{3}[526+572+2020] = \frac{1}{3}[3118]$$

= 1039.3 square units.

Example 17. Evaluate $\int_0^4 e^x dx$ by Simpson's rule, when $e = 2.72$, $e^2 = 7.39$, $e^3 = 20.09$, $e^4 =$ 54.60.

Sol. Given $y = e^x$, $a = 0$, $b = 4$, $h = 1$(As values of x varies in gap of 1).

x	0	1	2	3	4
$Y = e^x$	$e^0 = 1$	$e^1 = 2.72$	$e^2 = 7.39$	$e^3 = 20.09$	$e^4 = 54.6$

$y_1 = 1$, $y_2 = 2.72$, $y_3 = 7.39$, $y_4 = 20.09$, $y_5 = 54.6$

By Simpson's Rule

$$\int_0^4 e^x\,dx = \frac{h}{3}[(y_1+y_5)+2y_3+4(y_2+y_4)]$$

$$= \frac{1}{3}[(1+54.6)+2\times 7.39+4(2.72+20.09)]$$

$$= \frac{1}{3}[55.6+14.78+91.24] = 53.87\,.$$

EXERCISE - XI

1. Calculate by Simpson's rule an approximate value of $\int_0^1 \frac{dx}{1+x}dx$ by taking 10 equal interval.

2. Apply Simpson's rule to evaluate approximate value of $\int_4^8 \frac{dx}{x-3}$ by taking four equal intervals.

3. Find approximate area under the curve y = $(1 + x^2)$ when $0 \le x \le 4$ using Simpson's Rule by taking five ordinates.

4. Use Simpson's rule to approximate the area under the curve determined by the data gien below

x	1	2	3	4	5	6	7
y	15	27	30	33	20	19	4

ANSWERS

1. . 6931 2. 1.62 3. 25.33 4. 145

UNIT – 6

DIFFERENTIAL EQUATION

Learning Objectives

- To understand the basic concept about differential equations.
- To learn the solving technique for first order differential equations.

6.1 DIFFERENTIAL EQUATION

In engineering problems, mathematical models are made to represent certain problems. These mathematical models involve variables and derivatives of unknown functions. Such equations form the differential equation. Thus, a Differential Equation is defined as an equation in which y and its derivatives exists in addition to the independent variable x.

For example; (i) $\frac{dy^2}{dx^2} + 5y = 0$ (ii) $\frac{dy}{dx} + 3xy = 0$ (iii) $\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \frac{dy}{dx}$

In this chapter we will study only about ordinary differential equation.

Ordinary differential Equation: A differential equation in which dependent variable 'y' depends on only one variable 'x' i.e., y = f(x) . e.g.

$$\frac{dy}{dx} + y = 0$$

Partial differential Equation : A differential equation in which dependent variable 'z' depends on more than one variable say x and y i.e., z = f(x, y), For example,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

Order of differential equation : It is the order of the highest derivative appearing in the differential equation.

Example 1. Find the order (i) $\frac{d^2y}{dx^2}+5y=0$ (ii) $\frac{d^3y}{dx^3}+2\left(\frac{d^2y}{dx^2}\right)^2+3y=0$

Sol. (i) order is 2 (ii) order is 3

Degree of Differential Equation: It is the power of highest derivative appearing in a differential equation when it is free from radicals as far as derivatives are concerned fractional and negative powers.

Example 2. (i) $\left(\frac{d^2y}{dx^2}\right)^3+y=0$ (ii) $\frac{d^2y}{dx^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^3}$

Sol. Order – 2 Squaring both sides to remove fractional powers

degree – 3 $\left(\frac{d^2y}{dx^2}\right)^2=1+\left(\frac{dy}{dx}\right)^3$

Now order $\rightarrow$ 2, degree $\rightarrow$ 2

(iii) $\left(\frac{d^2y}{dx^2}\right)^2=\frac{1}{1+\frac{d^2y}{dx^2}}=\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{d^2y}{dx^2}\right)^3=1$ now order $\rightarrow$ 2, degree $\rightarrow$ 3

Linear Differential Equation : A differential equation in which dependent variable 'y' and its derivatives has power one and they are not multiplied together. When y = f(x)

Ex $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}+6\frac{dy}{dx}+y=0$

First Order Linear Differential Equation

A Differential Equation of the form

$$\frac{dy}{dx}+P(x)\,y=0 \qquad \text{where } y = f(x)$$

EXERCISE -I

1. Define the following with Examples

   (i) Differential equation

   (ii) Order of Differential Equation

   (iii) Degree of Differential Equation

2. Find the order and degree of the differential equation $\frac{d^3y}{dx^3} + \sqrt{\frac{d^2y}{dx^2}} + y = x$

   a. Order= 3, degree= 3
   b. Order= 3, degree= 2
   c. Order= 2, degree= 3
   d. Order= 3, degree= 1

3. Find the degree of the differential equation $\left(\frac{dy}{dx}\right)^2 = \frac{1}{1+\frac{d^2y}{dx^2}}$

   a. 1　　b. 2　　c. 4　　d. None of these

4. Find the degree of the differential equation $(x+1) = \left(\frac{d^2y}{dx^2}\right)^{\frac{2}{3}}$

   a. $\frac{2}{3}$　　b. 3　　c. 2　　d. 1

5. Determine the order and degree of the following differential equation. State whether the equations is linear or non linear

(i) $\left(\frac{d^3y}{dx^3}\right)^2 + 7\left(\frac{d^2y}{dx^2}\right)^{10} + 9\frac{dy}{dx} + 7y = 0$

(ii) $\frac{dy}{dx} + \cos x = 0$

(iii) $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y = 0$

(iv) $\frac{d^2y}{dx^2} + \log x\frac{dy}{dx} + y^2 = 0$

(v) $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 + 5y = 0$

(vi) $\left(\frac{d^2y}{dx^2}\right)^{\frac{1}{3}} = \left(\frac{dy}{dx}\right)^{\frac{1}{2}}$

(vii) $\frac{d^4y}{dx^4} + 3\frac{dy}{dx} = 1$

(viii) $\left[1+\left(\frac{d^2y}{dx^2}\right)^2\right]^{\frac{1}{3}} = \frac{d^3y}{dx^3}$

ANSWERS

2.(b) 3. (a) 4. (c)

5.	(i) Order – 3	degree –2	non linear
	(ii) Order – 1	degree – 1	Linear
	(iii) Order – 2	degree – 1	non linear
	(iv) Order – 2	degree – 1	non linear
	(v) Order – 3	degree – 1	non linear
	(vi) Order – 2	degree – 2	non linear
	(vii) Order – 4	degree – 1	linear
	(viii) Order – 3	degree – 3	non linear

Solution of a Differential Equation

The solution of differential equation is the relation between the variables involves in differential equation which satisfies the given differential equation.

In this chapter we will study the solution of first order differential equation by variable separable method.

Steps :

1. Separate the variables in given differential equation keeping in mind that dy and dx should be in numerator.

2. Now put the sign of $\int$ on both sides.

3. Integrate both sides separately by adding a constant on one side.

4. This will give us general sol of Differential equation.

Example 3. : Find the general solution of differential equation $\frac{dy}{dx} = \sin x$.

Sol : Given $\frac{dy}{dx} = \sin x$

Separate the variables, we get

$$dy = \sin x \, dx$$

Put the sign of $\int$ on both sides

$$\int dy = \int \sin x \, dx$$

$$y = -\cos x + C$$

Example 4. Find the general solution of differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$.

Sol : Separate the variables $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$

Put the sign of $\int$ on both sides

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$$\tan^{-1} y = \tan^{-1} x + c$$

Example 5. Find the solution of differential equation :

(i) $\frac{dy}{dx} = \frac{x}{y}$ (ii) $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$

(iii) $x dy - y \, dx = 0$ (iv) $\frac{dy}{dx} = \frac{1+y}{1+2x}$

Sol : (i) $\frac{dy}{dx} = \frac{x}{y}$ $\Rightarrow y dy = x \, dx$

$$\Rightarrow \int y \, dy = \int x \, dx \quad \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + c$$

(ii) $\frac{dY}{dx} = \frac{1}{\sqrt{1+x^2}}$ $\Rightarrow$ $dy = \frac{dx}{\sqrt{1+x^2}}$

$$\Rightarrow \quad \int dy = \int \frac{dx}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad y = \log|x+\sqrt{1+x^2}|+c$$

(iii) $x\,dy - y\,dx = 0 \Rightarrow \quad x\,dy = y\,dx$

$$\Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\Rightarrow \log y = \log x + c$$

(iv) $\dfrac{dy}{dx} = \dfrac{1+y}{1+2x} \qquad \Rightarrow \dfrac{dy}{1+y} = \dfrac{dx}{1+2x}$

$$\Rightarrow \quad \int \frac{dy}{1+y} = \int \frac{dx}{1+2x}$$

$$\Rightarrow \quad \log(1+y) = \frac{\log(1+2x)}{2} + c$$

EXERCISE-II

1. Solve the differential equation $\frac{dy}{dx} = 1 + x + y + xy$

   a. $y = x^2 + xy + x + c$ c. $ln(1+y) = x + \frac{x^2}{2} + c$

   b. $ln(1+x) = y + y^2 + c$ d. $(1+x)y = c$

2. The general sol of the differential equation $\frac{dy}{dx} = \frac{2-y}{x+1}$ is ....

   a. $log\left[\frac{2-y}{x+1}\right] = c$ c. $log(x+1)y = x + c$

   b. $log[(2-y)(x+1)]$=c   d. None of these

3. The general sol of the differential equation $\frac{dy}{dx} = e^x.e^{-y}$ is

   a. $e^y = e^x + c$ c. $e^{-y} = e^x + c$

   b. $e^x + e^{-y} = c$ d. $e^x e^y = c$

4. Solve the differential equations

(i) $\dfrac{dy}{dx} = \dfrac{x+2}{y}$ (iv) $\dfrac{dy}{dx} = \sqrt{\dfrac{1+y^2}{1+x^2}}$

(ii) $\frac{dy}{dx} = \frac{2-y}{x+1}$ (v) $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

(iii) $\frac{dy}{dx} = y \tan 2x$ (vi) $(1-x)y\,dx + (1+y)x\,dy = 0$

5. Solve the differential equation $\frac{dy}{dx} = 1 - x + y - xy$.

6. Solve the differential equation sin x cos y dx + cos x sin y dy = 0

7. Solve the differential equation $\frac{dy}{dx} = 2e^x y^3$.

ANSWERS

1. (c) 2. (b) 3. (a)

4. (i) $\frac{y^2}{2} = \frac{x^2}{2} + 2x + c$ (ii) $\frac{\log(2-y)}{-1} = \log(x+1) + c$

(iii) $\log y = \frac{\log \sec 2x}{2} + c$ (iv) $\log|y + \sqrt{1+y^2}| = \log|x + \sqrt{1+x^2}| + c$

(v) $e^y = e^x + \frac{x^3}{3} + c$ (vi) log xy = x − y + c

5. log (1 + y) = $x - \frac{x^2}{2} + c$ 6. log sec x + log sec y = c7. $-\frac{y^{-2}}{2} = 2e^x + c$

UNIT– 7

STATISTICS

Learning Objectives

- To understand the basic statistical measures like mean, mode, median and their different calculation methods.
- To learn about methods to calculate measure of mean derivatives, standard deviation, variance etc.

- To understand concept of correlation of data and different methods to calculate coefficients of correlation.

STATISTICS

Statistics is a branch of mathematics dealing with collection of data, analysis, interpretation, presentation and organization of data. In applying statistics into a problem, to for Example a scientific, industrial or social problem, it is conventional to begin with statistical population or a statistical model process to be studied in that problem.

Here in this chapter, we study about some measures of central tendency which are a central or typical values for a probability distribution. In other words, measures of central tendency are often called averages. Before we study about measures of central tendency, we must know about type of data.

Type of Data

a) Raw data : Data collected from source without any processing.

For example:The marks of ten students in math subject are 10,12,15,25,30,25,15,13,20,25

b) Discrete frequency distribution (Ungrouped data): The data which have not been divided into groups and presented with frequency (no. of times appeared) of data.

For example:The marks of ten students in math subject are given by

Marks(x):	10	12	13	15	20	25	30
Frequency(f):	1	1	1	2	1	3	1

c) Continuous frequency distribution (Grouped data): The data which have been divided into continuous group and presented with frequency (no. of times appeared) of individual group.

For example:The marks of students in math subject in a class are given by:

Marks class	10-20	20-30	30-40	40-50
Frequency	7	10	15	8

7.1 MEASURE OF CENTRAL TENDENCY

Measure of Central tendency are also classed as summary of statistics. In general,the mean, median and mode are all valid measure of central tendency.

<u>Mean</u>: It is average value of given data.

For Raw data;**Mean** $\bar{x} = \frac{\sum x_i}{n}$; where x_i is value of individual data

For Ungrouped and Grouped data; **Mean** $\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$; where n_i is the value of individual data and f_i is corresponding frequency of value x_i.

<u>Median</u> : It is middle value of arranged data.

For Grouped data; **Median =** $l + \frac{\left(\frac{n}{2} - c\right)}{f} * h$ where median class l = lower limit of class which has corresponding cumulative frequency equal to or greater than $\frac{N}{2}$median class,

$n = \Sigma f_i$,

c = Comulative frequency preceding to the median class cumulative frequency

h = length of interval,

f = frequency of median class

<u>Mode</u> : It is the term which appears maximum number of times in the given data.

OR

The term which has highest frequency in given data.

For Grouped data; **Mode** = $l + \frac{(f - f_1)}{2f - f_1 - f_2} * h$

Modal class = class which has maximum frequency value

where l = lower limit of modal class, h = length of interval,

f = frequency of modal class

f_1 = frequency of preceding class to modal class

f_2 = frequency of succeeding class to modal class

Example 1. Calculate mean for the following data.

21, 23, 25, 28, 30, 32, 46, 38, 48, 46

Sol : The given data is 21, 23, 25, 28, 30, 32, 46, 38, 48, 46

Total No. of observations = 10

Sum of observations = 21 + 23 + 25 + 28 + 30 + 32 + 46 + 38 + 48 + 46 = 337

$\therefore$ Mean = $\dfrac{Sum\, of\ observations}{Total\ No. of\ observations} = \dfrac{337}{10} = 33.7$

Example 2. Find the arithmetic mean of first 10 natural numbers.

Sol : First 10 natural numbers are 1,2,3,4,5,6,7,8,9,10

Sum of numbers = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55

No. of terms = 10

$\therefore$ Mean = $\dfrac{sum\, of\ numbers}{No. of\ terms} = \dfrac{55}{10} = 5.5$

Example3. Find the median of the daily wages of ten workers.

(Rs.) : 20, 25, 17, 18, 8, 15, 22, 11, 9, 14

Sol : Arranging the data in ascending order, we have

8, 9, 11, 14, 15, 17, 18, 20, 22, 25

Since there are 10 observations, therefore median is the arithmetic mean of $\left(\dfrac{10}{2}\right)^{th}$ and $\left(\dfrac{10}{2}+1\right)^{th}$ observations, so median = $\dfrac{15+17}{2} = 16$

Example 4. The following are the marks of 9 students in a class. Find the median.

34, 32, 48, 38, 24, 30, 27, 21, 35

Sol : Arranging the data in ascending order, we have

21, 24, 27, 30, 32, 34, 35, 38, 48

Since there are total 9 number of terms, which is odd. Therefore, median is the value of $\left(\frac{9+1}{2}\right)^{th}$ observation *i.e.* 32

Example 5. Find the mode from the following data:

110, 120, 130, 120, 110, 140, 130, 120, 140, 120

Sol : Arranging the data in the form of a frequency table, we have

Value:	110	120	130	140
Frequency :	2	4	2	2

Since the value 120 occurs the maximum number of times. Hence the mode value is 120

Example 6. The arithmetic mean of 7, 9, 5, 2, 4, 8, x is given to be 7. Find x.

Sol : $\bar{x} = \frac{7+9+5+2+4+8+x}{7}$

but $\bar{x}$ = 7

$$7 = \frac{35+x}{7}$$

$\Rightarrow$ 49 = 35 + x

$\Rightarrow$ x = 49 – 35 = 14

Example 7. Calculate range for the following data

19, 25, 36, 72, 51, 43, 28

Sol : Maximum value of given data = 72

Minimum value of given data = 19

Range of data = Maximum value – Minimum value

= 72 – 19 = 53

Example 8. Calculate arithmetic mean for the following :

Income (in Rs.)	:	500	520	550	600	800	1000
No. of Employees	:	4	10	6	5	3	2

Income (in Rs.) (x_i)	No. of Employees (f_i)	(f_ix_i)
500	4	2000
520	10	5200
550	6	3300
600	5	3000
800	3	2400
1000	2	2000
	$\Sigma f_i = 30$	$\Sigma x_if_i = 17900$

$$\therefore \quad \bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{17900}{30} = 590.67$$

Example 9. Calculate median for the following data

x_i	:	1	2	3	4	5	6	7	8	9
f_i	:	8	10	11	16	20	25	15	9	6

Sol :

x_i	f_i	C.f
1	8	8
2	10	18
3	11	29
4	16	45

Here N = 120

Now $\frac{N}{2} = \frac{120}{2}$ = 60

We find that the commutative frequency justgreater than $\frac{N}{2}$, is

5	20	65
6	25	90
7	15	105
8	9	114
9	6	120
	$=N = 120$ Σf_i	

65 and the value of x corresponding to 65 is ‘5’. Therefore median is ‘5’.

Example 10. Calculate mode for the following data:

x_i	91	92	96	97	101	103	108
f_i	3	2	3	2	5	3	3

Sol : As we know that mode is the highest frequency value and highest frequency is 5 and corresponding value is 101.

So mode value is 101

Example 11. Calculate mean for the following frequency distribution

Class interval	:	0-8	8-16	16-24	24-32	32-40	40-48
Frequency	:	8	7	16	24	15	7

Sol :

Class interval	Frequency(f_i)	Mid Value (x_i)	$f_i x_i$
0-8	8	4	32
8-16	7	12	84
16-24	16	20	320
24-32	24	28	672
32-40	15	36	540
40-48	7	44	308
	$\Sigma f_i = 77$		$\Sigma x_i f_i = 1956$

$$\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{1956}{77} = 25.40$$

Example 12. Calculate the median from the following distribution

Class :	5-10	10-15	15-20	20-25	25-30	30-35	35-40	40-45
Fre. :	5	6	15	10	5	4	2	2

Sol :

Class	f_i	C.f
5-10	5	5
10-15	6	11
15-20	15	26
20-25	10	36
25-30	5	41
30-35	4	45
35-40	2	47
40-45	2	49
	$\Sigma f_i = N = 49$	

We have N =49

Now $\frac{N}{2} = \frac{49}{2}$ = 24.5

commutative frequency just greater than $\frac{N}{2}$, is 24.5 and Corresponding class is 15-20

Thus, class 15-20 is the median class such that l = 15, f = 15

c.f. = 11, h = 5

$$\therefore \text{Median} = l + \frac{\left(\frac{N}{2} - C\right)}{f} \times h = 15 + \frac{(24.5 - 11)}{15} \times 5 = 15 + \frac{13.5}{3} = 15 + 4.5 = 19.5$$

Example 13. Calculate mode from the following data

Rent(in Rs.) :	20-40	40-60	60-80	80-100	100-120	120-140	140-160
No. of House :	6	9	11	14	20	15	10

Sol : By observation, we find that the highest frequency is 20

Hence 100-120 is the modal class.

$$\text{Mode} = l + \frac{(f - f_1)}{(2f - f_1 - f_2)} \times h$$

where l = 100, f = 20, f_1 = 14, f_2 = 15, h = 20

$\therefore$ Mode $= 100 + \dfrac{(20-14)}{2(20)-14-15} \times 20$

$$= 100 + \frac{6 \times 20}{11} = 100 + \frac{120}{11} = 100 + 10.91 = 110.91$$

7.2 MEASURE OF DISPERSION

Mean Deviation : The mean deviation is the first measure of dispersion. It is the average of absolute differences between each value in a set of value, and the average value of all the values of that set. Mean Deviation is calculated either from Mean or median. Here, we will study Mean Deviation about Mean

1. Mean deviation about Mean

(a) For Raw data

Mean Deviation about Mean = $\dfrac{\sum |x_i - \bar{x}|}{N}$

Where $\Sigma \to$ Represents summation

N $\to$ Number of terms or observations

Mean $\bar{x} = \dfrac{\sum x_i}{N}$

(b) For Ungrouped and Grouped data

Mean Deviation about Mean = $\dfrac{\sum f_i |x_i - \bar{x}|}{\sum f_i}$

Where $\Sigma \to$ Represents summation

$f_i \to$ frequency

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

Also Coefficient of Mean Deviation about mean is = $\dfrac{Mean\ deviation\ about\ mean}{Mean}$

2. Standard Deviation

It is the Root Mean Square value of deviations and also called in short form as R.M.S. value.

(a) For Raw data

$$\sigma = \text{Standard Deviation} = \sqrt{\frac{\sum (x_i - \bar{x})^2}{N}}$$

where $\bar{x} = \frac{\sum x_i}{N}$

N → Number of observations

(b) For Ungrouped and Grouped data

$$\sigma = \text{Standard Deviation} = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}}$$

where $\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$

<u>Some formulae</u> :

(i) Co-efficient Standard Deviation = $\frac{\text{Standard Deviation}}{\bar{x}}$

(ii) Variance = $(\sigma)^2$ = Square of Standard Deviation

(iii) Co-efficient of Variation = $\frac{S.D}{\bar{x}} \times 100$

Example 14. Calculate mean deviation about mean and its coefficient from the following data

21, 23, 25, 28, 30, 32, 46, 38, 48, 46

Sol:

x_i	$x_i - \bar{x}$	$\lvert x_i - \bar{x} \rvert$
21	– 12.7	12.7
23	–10.7	10.7
25	–8.7	8.7
28	–5.7	5.7
30	–3.7	3.7
32	–1.7	1.7
46	12.3	12.3
38	4.3	4.3
48	14.3	14.3
46	12.3	12.3
$\Sigma x_i = 337$		$\Sigma \lvert x_i - \bar{x} \rvert = 86.4$

$$\bar{x} = \frac{\Sigma xi}{N} = \frac{337}{10} = 33.7$$

$$M.D. = \frac{\sum |x_i - \bar{x}| f_i}{\sum f_i} = \frac{86.4}{10} = 8.64$$

Coefficient of M.D. = $\frac{M.D.}{\bar{x}} = \frac{8.64}{33.7} = 0.26$

Example 15. Find mean deviation about mean of the following data

x_i	:	3	5	7	9	11	13
f_i	:	2	7	10	9	5	2

Sol :

x_i	f_i	$x_i f_i$	$x_i - \bar{x}$	$\lvert x_i - \bar{x}\rvert$	$\lvert x_i - \bar{x}\rvert f_i$
3	2	6	–4.8	4.8	9.6
5	7	35	–2.8	2.8	19.6
7	10	70	–0.8	0.8	8.0
9	9	81	1.2	1.2	10.8
11	5	55	3.2	3.2	16.0
13	2	26	5.2	5.2	10.4
	$\Sigma f_i = 35$	$\Sigma x_i f_i = 273$			$\Sigma \lvert x_i - \bar{x}\rvert f_i = 74.4$

$$\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{273}{35} = 7.8 \qquad M.D. = \frac{\sum |x_i - \bar{x}| f_i}{\sum f_i} = \frac{74.4}{35} = 2.13$$

Example 16. Find mean deviation about the mean for the following data

Marks obtained	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Number of students	2	3	8	14	8	3	2

Sol :

Marks obtained	No. of students(f_i)	Mid point(x_i)	$f_i x_i$	$x_i - \bar{x}$	$\lvert x_i - \bar{x}\rvert$	$\lvert x_i - \bar{x}\rvert f_i$
10-20	2	15	30	–30	30	60
20-30	3	25	75	–20	20	60
30-40	8	35	280	–10	10	80
40-50	14	45	630	0	0	0
50-60	8	55	440	10	10	80
60-70	3	65	195	20	20	60
70-80	2	75	150	30	30	60

$\Sigma f_i = 40$		$\Sigma x_i f_i = 1800$		$\Sigma \lvert x_i - \overline{x} \rvert f_i = 400$

$$\overline{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{1800}{40} = 45 \qquad M.D. = \frac{\sum \lvert x_i - \overline{x} \rvert f_i}{\sum f_i} = \frac{400}{40} = 10$$

Example 17. Calculate deviation and variance of the following data :

6, 8, 10, 12, 14, 16, 18, 20, 22, 24

Sol :

x_i	$x_i - \overline{x}$	$(x_i - \overline{x})^2$
6	–9	81
8	–7	49
10	–5	25
12	–3	9
14	–1	1
16	1	1
18	2	4
20	5	25
22	7	49
24	9	81
$\Sigma x_i = 150$		$\Sigma (x_i - \overline{x})^2 = 330$

$$\overline{x} = \frac{\sum x_i}{n} = \frac{150}{10} = 15$$

$$S.D. = \sqrt{\frac{\sum (x_i - \overline{x})^2}{n}} = \sqrt{\frac{330}{10}} = \sqrt{33}$$

Variance = $(S.D.)^2 = \left(\sqrt{33}\right)^2 = 33$

Example 18. Find variance and co-efficient of variation for the following data

x_i	4	8	11	17	20	24	32

f_i	3	5	9	5	4	3	1

Sol :

x_i	f_i	$f_i x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
4	3	12	–10	100	300
8	5	40	–6	36	180
11	9	99	–3	9	81
17	5	85	3	9	45
20	4	80	6	36	144
24	3	72	10	100	300
32	1	32	18	324	324
	$\Sigma f_i = 30$	$\Sigma x_i f_i = 420$			$\Sigma f_i(x_i - \bar{x})^2 = 1374$

$$\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{420}{30} = 14$$

$$\sigma = \quad S.D = \sqrt{\frac{\sum f_i(x_i - \bar{x})^2}{\sum f_i}} = \sqrt{\frac{1374}{30}} = \sqrt{45.8} = 6.77$$

Variance = $(\text{S.D.})^2 = 45.8$

Coefficientof variation = $\dfrac{S.D}{\bar{x}} \times 100 = \dfrac{6.77}{14} \times 100 = 48.35$

Example 19. Calculate the mean, variance and coff. of S.D.for the following distribution:

Class	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	3	7	12	15	8	3	2

Sol :

Class	f_i	x_i	$f_i x_i$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
30-40	3	35	105	729	2187
40-50	7	45	315	289	2023
50-60	12	55	660	49	588
60-70	15	65	975	9	135

70-80	8	75	600	169	1352
80-90	3	85	255	529	1587
90-100	2	95	190	1089	2178
	50		3100		10050

$$\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{3100}{50} = 62$$

Variance = $\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i} = \frac{10050}{50} = 201$

S.D = $\sqrt{201} = 14.18$

Coff. of S.D = $\frac{S.D}{\bar{x}} = \frac{14.18}{62} = 0.22$

EXERCISE – I

1. Mean of first 10 natural numbers is
   a. 5.5 b. 5 c. 4 d. 10
2. Median value of first 20 natural numbers is
   a. 10 b. 10.5 c. 11 d. 12
3. Mode of following data : 2,2,4,4,4,5,6,7,10,11 is
   a. 2 b. 4 c. 10 d. 11
4. Calculate the mean of 1, 2, 3, 4, 5, 6
5. Calculate the median of the data 13, 14, 16, 18, 20, 22
6. Median of the following observations 68, 87, 41, 58, 77, 35, 90, 55, 92, 33 is 58. If 92 is replaced by 99 and 41 by 43, then find the new median.
7. Find the mode of the set of values

   2.5, 2.3, 2.2, 2.4, 2.2, 2.7, 2.7, 2.5, 2.3, 2.3, 2.6
8. Find median of the series 3, 6, 6, 9, 12, 10
9. Find median of the series 4, 8, 6, 12, 15
10. Find A.M. for the following

x	:	10	11	12	13	14	15
f	:	2	6	8	6	2	6

11. Find the value of median for the following data

x	5	6	7	8	9	10	11	12	13	15	18	20
f	1	5	11	14	16	13	10	70	4	1	1	1

12. Find the mean deviation about mean of daily wages (in Rs.) of 10 workers : 13, 16, 15, 15, 18, 15, 14, 18, 16, 10 also find its co-efficient.

13. Find M.D. about mean of the following data

x	:	3	5	7	9	11	13
f	:	2	7	10	9	5	2

14. Find the mean deviation about the mean for the following data.

Income per day	0-100	100-200	200-300	300-400	400-500	500-600	600-700	700-800
Number of Persons	4	8	9	10	7	5	4	3

15. Find mean deviation about mean for the following data

x_i	5	10	15	20	25
f_i	7	4	6	3	5

16. Find mean deviation for 4, 7, 8, 9, 10, 12, 13, 17

17. Find standard deviation for the following data

x_i	3	8	13	18	23
f_i	7	10	15	10	6

18. Calculate mean, variance and standard deviation for the following data

Cass Interval	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	3	7	12	15	8	3	2

19. Find mean and variance for 6, 7, 10, 12, 13, 4, 8, 12

20. Find variance for the following frequency distribution

Cass Interval	0-30	30-60	60-90	90-120	120-150	150-180	180-210
Frequency	2	3	5	10	3	5	2

21. Find mean for the following data

Class interval	0-10	10-20	20-30	30-40	40-50
Frequency	5	8	15	16	6

ANSWERS

(1) a	(2) b	(3) b	(4) 7/2	(5) 17	(6) 58
(7) 2.2	(8) 7.5	(9) 8	(10) 12.6	(11) 12	(12) 1.6
(13) 2.13	(14) 157.92	(15) 6.32	(16) 3	(17) 6.12	

(18) Mean; 62 Variance: 201 SD: 14.18 (19) Mean: 9 Variance: 9.25

(20) 2276 (21) 64

7.3 CORRELATION AND RANK OF CORRELATION

Correlation : If two quantities are in such way that changes in one leads to change in other then we say that these two quantities are correlated to each other. For Example, the age of a person and height of the person are correlated. If this relation exist in two variables then we say simple correlation. In this chapter, we study about simple correlation

Types of Correlation

(i) Positive Correlation : Two quantities are positively correlated if increase in one leads to increase in other or decrease in one leads to decrease in other, then we say two quantities are positively correlated. For example

Year	:	1960	1970	1980	1990	2000
Population in village	:	1250	1390	1490	1670	1700

(ii) Negative Correlation : Two quantities are negatively correlated if increase in one leads to decrease in other or vice versa then we say the two quantities are negatively correlated. For example

Year	:	1960	1970	1980	1990	2000
No. of trees in a town	:	2000	1730	1600	1520	990

Correlation Coefficient : The numerical measure of degree of relationship correlation between two quantities is called correlation coefficient.

Method of Studying Correlation

(i) Scatter Diagram Method
(ii) Graphic Method
(iii) Karl Pearson's Coefficient of Correlation
(iv) Rank Correlation Method

Here we will study only about Rank Correlation Method.

Rank Correlation Coefficient: In this method, firstly ranking are given to the observations then, co-efficient of rank correlation is given as :

$$r = 1 - \frac{6\sum d^2}{N(N^2 - 1)}$$

where N $\rightarrow$ Number of observations

d $\rightarrow$ difference of ranks

Here r lies between -1 and 1

(i) r > 0 and nearly equal 1 means two quantities are positively correlated
(ii) r < 0 means two quantities are negatively correlated

Example 20. Calculate co-efficient of rank correlation between X and Y from the following data:

X	:	45	70	65	30	90	40	50	75	85	60
Y	:	35	90	70	40	95	45	60	80	30	50

Sol:

X	Y	R_1	R_2	$d = R_1 - R_2$	d^2
45	35	8	9	-1	1
70	90	4	2	2	4
65	70	5	4	1	1
30	40	10	8	2	4
90	95	1	1	0	0
40	45	9	7	2	4
50	60	7	5	2	4
75	80	3	3	0	0

85	30	2	10	-8	64
60	50	6	6	0	0
					$\Sigma d^2 = 82$

$$r = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6 \times 82}{10(99)} = 1 - \frac{492}{990} = 1 - 0.49 = 0.51$$

Example 21. In a fancy-dress competition, two judges accorded following ranks to the 10 participants:

Judge X	1	2	3	4	5	6	7	8	9	10
Judge Y	10	6	5	4	7	9	8	2	1	3

Calculate the co-efficient of rank correlation.

Sol:

Judge X = R_1	Judge Y = R_2	d = R_1-R_2	d^2
1	10	-9	81
2	6	-4	16
3	5	-2	4
4	4	0	0
5	7	2	4
6	9	3	9
7	8	-1	1
8	2	6	36
9	1	8	64
10	3	7	49
			$\Sigma d^2 = 264$

$$r = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6 \times 264}{10(99)} = 1 - \frac{1584}{990} = 1 - 1.6 = -0.6$$

Example 22. The co-efficient of rank correlation between X and Y is 0.143. If the sum of scourers of the differences is 48, find the value of N.

Sol : r = 0.143, $\Sigma d^2 = 48$, N = ?

$$r = 1 - \frac{6\sum d^2}{N(N^2-1)} \quad or \quad 0.143 = 1 - \frac{6 \times 48}{N(N^2-1)}$$

$$0.143 - 1 = \frac{-288}{N(N^2-1)} \quad or \quad -0.857 = \frac{-288}{N(N^2-1)}$$

$$N(N^2) - 1 = \frac{228}{0.857}$$

$$N(N^2 - 1) = 336$$

By hit and trial method, we get n = 7

i.e. $7(7^2 - 1) = 7(49-1) = 7(48) = 336$

So n = 7

EXERCISE – II

1. There are two sets X and Y of data. If increase in X values leads to decrease in corresponding values of Y set, then
   a. X and Y are positively correlated
   b. X and Y are negatively correlated
   c. X and Y are not correlated
   d. X and Y are equal sets
2. Formula for Rank correlation is:
   a. $r = 1 - \frac{6\sum d^2}{N(N^2-1)}$ c. $r = 1 + \frac{6\sum d^2}{N(N^2-1)}$
   b. $-1 \le r \le 1$ d. $r > 0$
3. The value of rank correlation coefficients satisfies
   a. $-1 \le r \le 1$ b. $r > 1$ c. $r < -1$ d. $r \to \infty$
4. Formula for mean deviation about mean for frequency data is given as
   a. $\frac{\sum fi|xi - \bar{x}|}{\sum fi}$ b. $\frac{\sum fi|xi + \bar{x}|}{\sum fi}$ c. $\frac{\sum fi(xi - \bar{x})}{\sum fi}$ d. $\frac{\sum fi(xi - \bar{x})^2}{\sum fi}$
5. Fill in the blanks
   a. Variance is .... of standard deviation
   b. Coefficient of variation = ... × 100
6. The ranking of ten students in two subjects A and B are

A :	3	5	8	4	7	10	2	1	6	9
B :	6	4	9	8	1	2	3	10	5	7

7. Ten students got the following marks in Mathematics and Physics

Marks in Maths :	78	36	98	25	75	82	90	62	65	39
Marks in Phy. :	84	51	91	60	68	62	86	58	53	47

8. Calculate the co-efficient of rank correlation between X and Y from the following data:

X :	10	12	18	16	15	19	13	17
Y :	30	35	45	44	42	48	47	46

9. The sum of squares of differences on the ranks of n pairs of observations is 126 and the co-efficient of rank correlation is -0.5. Find n.
10. The rank correlation co-efficient between marks obtained by some students in statistics and Economics is 0.8. If the total of squares of rank difference is 33. Find the no. of students.

ANSWERS

1. b 2. a 3. a 4. a 5.a. Square b. $\frac{\text{Standard deviation}}{\bar{x}}$

6. -0.297 7.0.82 8. 0.74 9. n = 8 10. 10

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